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THE DIFFRACTION OF SOUND PULSES BY AN INFINITELY LONG STRIP

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The solution is obtained for the two-dimensional diffraction problem of a perfectly-reflecting strip subjected to a plane sharp-fronted pulse of constant unit pressure at normal incidence.

Numerical calculations show in particular that the equalization of pressure round the strip 'overshoots' to produce pressures on the back of the strip up to a maximum of 21 % in excess of the incident unit pressure, with pressures correspondingly below incident pressure on the front of the strip. The calculations also indicate that the pressure has become effectively steady, to within 3 % or less, at the incident unit pressure in a time $5b/c$ after the pulse strikes the strip, where $2b$ is breadth of strip and c is velocity of sound.

The extension to any shape of normally incident plane pulse is given in terms of our basic solution by simple application of the principle of superposition. The formal extension of the solution to the general case of any incident two-dimensional pulse field is also given.

Finally, it is noted that the same general method can be applied to a number of related two-dimensional diffraction problems and in particular, solutions and numerical results have been thus obtained for the problems of a slit and a regular grating. It is proposed to consider these problems in further papers.

1. Introduction

The diffraction of sound waves and sound pulses by a perfectly-reflecting half-plane was originally discussed by Sommerfeld (1896, 1901) and subsequently by Macdonald (1902) and Lamb (1906, 1910). For the wave case, interest in the problem has been revived by Magnus (1941) who showed that it can be reduced mathematically to the consideration of an integral equation, which he then solves by a somewhat roundabout analysis to obtain Sommerfeld's solution. Recently Copson (1946*a*) has given a more direct and elegant solution of this integral equation.

Friedlander (1946), has considered the pulse problem in some detail and gives some interesting results of numerical calculations for different shapes of pulse.

When we proceed from the half-plane case to those of an infinite strip, an infinite slit or a regular grating, we pass to cases where the obstacle introduces a finite dimension and the problem becomes more complicated. For an incident sinusoidal wave-train we have, however, the classical solutions of Rayleigh (see Lamb 1932, pp. 517–538) for the extreme case of wave-length long compared with obstacle or slit dimensions. Such solutions have also been recently derived by Copson (1946*b*) as approximate solutions of integral equations. The problems of the strip and the slit subjected to an incident wave-train can further be solved exactly in the form of an infinite series involving Mathieu functions as discussed by Sieger (1908) and by Morse & Rubinstein (1938).

Friedlander (1946) mentions a suggestion by Bullard that Huygens's principle, in the form of Kirchoff's solution (Jeans 1925, p. 522) of the wave-equation, can be applied to pulse diffraction, but points out that the physical assumptions necessary for such application to lead directly to an explicit solution, are not usually satisfied by sound pulses incident on normal obstacles. However, in some diffraction problems, reasonable approximations to the physical conditions can be made by which Kirchoff's solution of the wave-equation leads to a simple type of integral equation which can then be solved either formally or by a numerical process. In particular, it is relevant to note that in one such problem Kirkwood (unpublished) obtained the solution of the integral equation by use of the Laplace transformation.

Finally, the author has now obtained solutions for the two-dimensional diffraction problems of a plane sound pulse incident normally on (i) a perfectly-reflecting strip, (ii) a slit in a perfectly-reflecting plane, (iii) a regular grating of perfectly-reflecting strips, and has carried out some numerical calculations to derive physical conclusions.

The first two basic steps are the derivation of an integral equation by use of Huygens's principle and its transformation, to a more amenable integral equation, by the Laplace transformation. The third basic step involves the solution of the transformed equation by using the known solution of the half-plane problem.

In the present paper is given the derivation of the solution for the strip problem and a discussion of some physical implications of the results of numerical calculation. In subsequent papers the solutions and results for the slit and grating problems will be similarly considered. To avoid undue repetition in these later papers, some initial formulae are derived under fairly general conditions and we then proceed to the specific problem of the strip (figure 1) subjected to a normally incident pulse of the basic $H(t)$ shape (figure 2).

The analysis is done throughout in terms of pressure rather than velocity-potential as the former is usually the quantity of physical interest in pulse problems. Mathematically, of course, this choice is immaterial since both quantities satisfy the wave-equation and the same form of boundary condition at the strip.

2. General theorems

We consider first the general case, not necessarily two-dimensional, of an infinite perforated screen occupying the plane $x = 0$ and containing apertures of arbitrary shape and size, the screen itself being perfectly reflecting. Let this screen be subjected to pulses from any combination of point sources lying to the right ($x > 0$) of the screen. In particular, this includes a plane pulse as the limiting case of a pulse from a source at infinite distance. Let $p_i(t, x, y, z)$ be the total pressure at any point (x, y, z) and time t due to all the sources in the absence of the screen. We shall refer to p_i as the 'incident field of pressure' or more briefly as 'incident field' or 'incident pressure'. Let $p(t, x, y, z)$ similarly denote the pressure in the actual problem when the screen is present.

Since the problem is linear, we can superimpose the corresponding mirror image problem in which an incident field $p_i(t, -x, y, z)$ arrives from the left and the pressure is $p(t, -x, y, z)$ at any point. But in the combined problem the boundary condition $\partial p/\partial n = 0$ over front and back of the screen is automatically satisfied by the two incident fields; i.e. we can remove the screen without altering the boundary conditions. Hence the pressure anywhere in the combined problem is simply

$$p(t, x, y, z) + p(t, -x, y, z) = p_i(t, x, y, z) + p_i(t, -x, y, z). \tag{1}$$

We thus have in our original problem that the sum of the pressures at any point P, (x, y, z) and at its mirror image P', $(-x, y, z)$ in the screen is simply equal to the pressure at P due to complete reflexion of the incident field at the plane $x = 0$. This means, in particular, that if we can determine the field to the rear we can immediately deduce the field in front of the screen and conversely.

If we now let $x \to 0$ in equation (1) and if the point $(0, y, z)$ lies in an aperture of the screen then, since the pressure is continuous across the aperture we obtain in the limit, after dividing by two,

$$p = p_i \tag{2}$$

for the pressure at any point within an aperture.

On the other hand, if $(0, y, z)$ lies on the screen itself, the pressure is not necessarily equal at back and front and when $x \to 0$ in equation (1) we obtain

$$p_f + p_b = 2p_i \tag{3}$$

for any point on the screen itself, where the suffixes f and b refer to front and back of the screen.

Now, for a point $(-X, y, z)$, where $X > 0$, to the rear of the screen we can apply the well-known Kirchoff solution (Jeans 1925, p. 522)

$$p(t, -X, y, z) = -\frac{1}{4\pi}\iint\left\{\frac{1}{cr}\frac{\partial r}{\partial n}\frac{\partial p}{\partial t} - p\frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial p}{\partial n}\right\}_{t-r/c} dS. \tag{4}$$

Where c is the velocity of sound, r is distance from our point to any point on the surface S of inward normal n and the integrand is to be evaluated at time $t - r/c$. For our surface S we can choose the plane $x = -\epsilon$, where $X > \epsilon > 0$, and a large semi-circle of radius R to the left of this plane and we choose our origin of time so that the incident field first arrives at the origin $(0, 0, 0)$ at time $t = 0$.

Since our incident field is assumed to be arriving from the right of the screen and the semicircle is to the left we can, for any finite t, X, y, z by choosing R large enough, ensure that $t - r/c$ on the semicircle corresponds to times prior to the arrival of any field. The contribution to the surface integral in equation (4) can thus be taken solely as arising from the plane $x = -\epsilon$ and if we now let $\epsilon \to 0$ the surface integral reduces to one over the apertures and back to the screen.

The argument (Jeans 1925, p. 522) leading to the Kirchoff solution of the wave-equation introduces a converging wave at a point within the region bounded by S. If instead we introduce a wave converging to a point (x', y', z') outside this region and otherwise follow the same argument we obtain a result similar to equation (4) but with zero on the left-hand side, namely,

$$0 = -\frac{1}{4\pi}\iint\left\{\frac{1}{cr'}\frac{\partial r'}{\partial n}\frac{\partial p}{\partial t} - p\frac{\partial}{\partial n}\left(\frac{1}{r'}\right) + \frac{1}{r'}\frac{\partial p}{\partial n}\right\}_{t-r'/c} dS, \tag{5}$$

where r' is the distance from (x', y', z') to any point of S.

We can now apply equation (5) to the same surface, $x = -\epsilon$ and large semicircle to left of screen, as before and choose for $(x'y'z')$ the mirror image (X, y, z) in the screen of our previous point $(-X, y, z)$ to the rear of the screen. As before, when $R \to \infty$ we get no contribution from the large semicircle while as $\epsilon \to 0$ the surface integral reduces to one over

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the rear of the screen and the apertures. But for the image points we then have the simple relations

$$\left.\begin{aligned} r' &= r, \\ \frac{\partial r'}{\partial n} &= -\frac{\partial r}{\partial n}, \end{aligned}\right\} \tag{6}$$

so that the last term on the right of equation (5) becomes identical with the last term on the right of equation (4) while the first two terms on the right of equation (5) are equal but of opposite sign to the similar terms in equation (4). Hence adding equations (4) and (5) we obtain

$$p(t, -X, y, z) = -\frac{1}{2\pi}\iint\left(\frac{1}{r}\frac{\partial p}{\partial n}\right)_{t-r/c} dS \quad (X>0), \tag{7}$$

whilst subtracting equation (5) from equation (4) we obtain the alternative expression

$$p(t, -X, y, z) = -\frac{1}{2\pi}\iint\left\{\frac{1}{cr}\frac{\partial r}{\partial n}\frac{\partial p}{\partial t} - p\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right\}_{t-r/c} dS \quad (X>0). \tag{8}$$

The surface integrals in both equations (7) and (8) are taken over the apertures and the back of the screen. Since $\partial p/\partial n = 0$ on the screen, however, the surface integral in equation (7) is effectively taken over the apertures only.

Conversely, we can transform equation (8) so that it involves a surface integral over the back of the screen only. Thus equation (8) applies for any arbitrary distribution of apertures and in particular to the limiting case of all aperture and no screen for which $p = p_i$ everywhere. Thence, by subtraction, equation (8) will hold with $p_i - p$ for p, namely

$$p_i(t, -X, y, z) - p(t, -X, y, z) = -\frac{1}{2\pi}\iint\left\{\frac{1}{cr}\frac{\partial r}{\partial n}\frac{\partial}{\partial t}(p_i - p) - (p_i - p)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right\}_{t-r/c} dS. \tag{9}$$

But from equation (2), $p_i - p = 0$ over the apertures and thus the surface integral in equation (9) involves in effect only integration over the back of the screen itself.

Summing up, by virtue of equations (1), (8) and (9) a complete formal explicit solution of the problem can be obtained if we can determine either (*a*) the distribution of $\partial p/\partial n$ over the apertures, or (*b*) the distribution of pressure on the back of the screen, with varying time t.

The relations we have given in this section are, of course, the pulse analogues of well-known relations (Lamb 1932, chapter x) for sinusoidal waves.

3. Plane $H(t)$ pulse normally incident on strip

3·1. *Derivation of integral equation*

We now proceed to consider the particular problem of figure 1 in which an infinite strip of finite width is struck at normal incidence by a plane pulse. Without any real loss of generality we take the time variation in the incident pulse to be given by Heaviside's unit function $H(t)$ as in figure 2 since the formal solution for any other shape of pulse is then given immediately by superposition (§ 3·8).

For ease in writing formulae we shall also make the problem non-dimensional by choosing the width $2b$ of the strip as our unit of length and the time $2b/c$ for sound to travel this width as our unit of time. The wave velocity is then unity and our incident pulse is

$$p_i = H(t+x), \tag{10}$$

where time is measured from the instant at which the pulse front strikes the strip. The problem is two-dimensional with all quantities independent of z and we take the strip to be in the plane $x = 0$ but leave our origin for y temporarily unspecified.

For any point $(-X, y, 0)$, where $X > 0$, to the rear of the strip (though not necessarily in the shadow) the general relation (7) becomes (noting $c = 1$),

$$p(t, -X, y) = -\frac{1}{2\pi}\iint\left(\frac{1}{r}\frac{\partial p}{\partial n}\right)_{t-r} dS, \tag{11}$$

where the surface integral is taken over the rear of the plane $x = 0$.

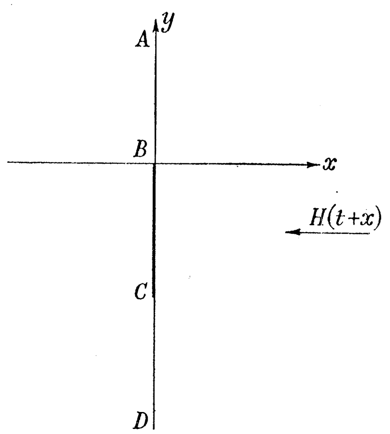

FIGURE 1

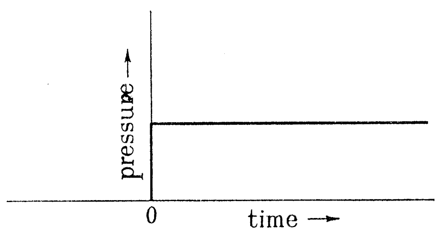

FIGURE 2. Incident pulse.

If we now write in the integrand

$$\frac{\partial p}{\partial n} = -\left(\frac{\partial p}{\partial x}\right)_{x=0} = -\psi(y_0, t) \tag{12}$$

for a point $(0, y_0, z_0)$ at time t we obtain

$$p(t, -X, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{r}\psi(y_0, t-r)\, dy_0\, dz_0 \quad (X > 0), \tag{13}$$

where

$$r^2 = X^2 + (y-y_0)^2 + z_0^2. \tag{14}$$

But this holds in the limit $X \to 0$ for a point in the aperture where equation (2) also holds and hence from equations (2), (10) and (13) we obtain

$$H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{r}\psi(y_0, t-r)\, dy_0\, dz_0, \tag{15}$$

where

$$r^2 = (y-y_0)^2 + z_0^2 \tag{16}$$

for any point $(0, y, 0)$ lying on AB and CD in figure 1.

We now introduce the Laplace transformation by writing

$$\chi(y_0, \lambda) = \int_0^{\infty}\psi(y_0, t)\, e^{-\lambda t}\, dt, \tag{17}$$

and since we assume an initial state of rest prior to the arrival of the incident pulse at $t = 0$ we have

$$\psi(y_0, t) \equiv 0 \quad (t < 0), \tag{18}$$

whence it follows that
$$\int_0^\infty e^{-\lambda t}\psi(y_0, t-r)\,dt = \int_r^\infty e^{-\lambda t}\psi(y_0, t-r)\,dt$$
$$= \int_0^\infty e^{-\lambda(t'+r)}\,\psi(y_0, t')\,dt'$$
$$= e^{-\lambda r}\chi(y_0, \lambda). \tag{19}$$

Hence if we apply the Laplace transformation to equation (15) we obtain

$$\frac{1}{\lambda} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\lambda r}}{r}\chi(y_0, \lambda)\,dy_0\,dz_0, \tag{20}$$

where r is given by equation (16).

We can now perform the z_0 integration by changing from z_0 to r as variable of integration to obtain
$$\frac{1}{\lambda} = \frac{1}{\pi}\int_{-\infty}^{\infty}\chi(y_0, \lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0, \tag{21}$$

where K_0 denotes, as usual, Macdonald's Bessel function of order zero.

We now choose the origin of co-ordinates to be at the upper edge B of the strip and since $\partial p/\partial n = 0$ on the strip itself we have from equations (12) and (17)

$$\chi(y_0, \lambda) = 0 \quad (0 \geqslant y_0 \geqslant -1), \tag{22}$$

whilst by symmetry about the centre of the strip we have

$$\chi(y_0, \lambda) = \chi(-1-y_0, \lambda) \quad (-1 \geqslant y_0). \tag{23}$$

Inserting equations (22) and (23) in equation (21) we then obtain

$$\frac{1}{\lambda} = \frac{1}{\pi}\int_0^\infty \chi(y_0, \lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 + \frac{1}{\pi}\int_{-\infty}^{-1}\chi(-1-y_0, \lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 \quad (y \geqslant 0). \tag{24}$$

In this equation y_0 is now purely a variable of integration and we can change this variable in the second integral by writing y_0 for $-1-y_0$ to give a positive range of integration; for a point $(0, y)$ on AB (figure 1) we can then write the equation in the form

$$\pi = \lambda\int_0^\infty \chi(y_0, \lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 + \lambda\int_0^\infty \chi(y_0, \lambda)\,K_0\{\lambda(1+y+y_0)\}\,dy_0 \quad (y \geqslant 0), \tag{25}$$

where we note that the modulus sign can now be dropped in the second integral.

Equation (24) holds, of course, also for a point $(0, y)$, $y \leqslant -1$, on CD in figure 1 but since we have used symmetry already in obtaining equation (24) its application to a point on CD, after change of variables, will only lead mathematically to the same equation (25) obtained by considering a point on AB.

3·2. *Solution of integral equation*

Equation (25) forms the basic integral equation of our problem and is analogous to the simpler equation considered by Magnus (1941) and Copson (1946*a*) for the half-plane subjected to a sinusoidal train of waves.

To commence the solution of equation (25) we note that the second integral in it corresponds to the contribution from CD for a point on AB and if we omit this contribution we have in fact the integral equation for the half-plane case. Thus, if we denote by χ_0 the solution for the half-plane case, then χ_0 will satisfy the integral equation

$$\lambda\int_0^\infty \chi_0(y_0,\lambda)\, K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 = \pi, \tag{26}$$

and in the known results for the half-plane case as given by Friedlander (1946) we have merely to substitute our assumed form (10) for the incident wave and apply the Laplace transformation to obtain

$$\chi_0(y,\lambda) = 1+\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-\lambda y \sec^2\theta}\tan^2\theta\,d\theta, \tag{27}$$

where we may note that χ_0 is a function only of the product λy.

We now seek a solution of the equation (25) for the strip case by the method of successive substitutions in which the first integral is regarded as the dominant term on the right-hand side of the equation. We therefore write

$$\chi = \chi_0-\chi_1+\chi_2-\chi_3\ldots(-1)^r\chi_r\ldots, \tag{28}$$

where χ_0 is given by equation (27) whilst

$$\int_0^\infty \chi_{r+1}(y_0,\lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 = \int_0^\infty \chi_r(y_0,\lambda)\,K_0\{\lambda(y+y_0+1)\}\,dy_0 \quad (r\geqslant 0,\ y\geqslant 0). \tag{29}$$

Since we know χ_0, equation (29) may be regarded as an integral equation in which the right-hand side is a known function of y and λ and an explicit solution for χ_{r+1} is required.

This equation (29) is, however, a generalized version of the integral equation (26) for the half-plane in which a general function of y and λ appears on the right-hand side. Let us therefore consider generally the integral equation

$$\lambda\int_0^\infty u(y_0,\lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 = v(y,\lambda) \quad (y\geqslant 0) \tag{30}$$

in which v is a known function of y whilst u has to be determined for $y\geqslant 0$.

Now for positive values of y we know that χ_0, as given by equation (27), satisfies equation (26) whilst for negative values of y we find by substitution from equations (27) and (139) and integration that

$$\left.\begin{aligned}\lambda\int_0^\infty \chi_0(y_0,\lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 &= 2\int_0^{\frac{1}{2}\pi} e^{\lambda y\sec^2\theta}\,d\theta\\ &= M(-\lambda y)\end{aligned}\right\}\quad (y\leqslant 0) \tag{31}$$

which defines a known function M.

We now replace y in equation (30) by $y+y_1$, where $y\geqslant 0$, $y_1\geqslant 0$, and subtract the resulting equation from equation (30) to obtain

$$\lambda\int_0^\infty u(y_0,\lambda)\,[K_0\{\lambda\,|\,y-y_0\,|\}-K_0\{\lambda\,|\,y+y_1-y_0\,|\}]\,dy_0 = v(y,\lambda)-v(y+y_1,\lambda) \quad (y\geqslant 0, y_1\geqslant 0). \tag{32}$$

We then multiply this equation (32) by $\chi_0(y,\lambda)\,dy$ and integrate from $y=0$ to ∞ to give

$$\begin{aligned}\lambda\int_0^\infty \chi_0(y,\lambda)\int_0^\infty u(y_0,\lambda)\,[K_0\{\lambda\,|\,y-y_0\,|\}-K_0\{\lambda\,|\,y+y_1-y_0\,|\}]\,dy_0\,dy\\ =\int_0^\infty \chi_0(y,\lambda)\,\{v(y,\lambda)-v(y+y_1,\lambda)\}\,dy\\ = W(y_1,\lambda) \quad (y_1\geqslant 0),\end{aligned} \tag{33}$$

where W is known since χ_0 and v are known.

But from equations (26) and (31) we have

$$\left.\begin{aligned}\lambda\int_0^\infty \chi_0(y,\lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy &= \pi \quad (y_0\geqslant 0),\\ \lambda\int_0^\infty \chi_0(y,\lambda)\,K_0\{\lambda\,|\,y+y_1-y_0\,|\}\,dy &= \pi \quad (y_0\geqslant y_1)\\ &= M\{\lambda(y_1-y_0)\} \quad (y_0\leqslant y_1).\end{aligned}\right\} \tag{34}$$

Hence, assuming that the solution u is such that the order of integration can be changed in equation (33) and applying equation (34), we obtain

$$\int_0^{y_1} u(y_0,\lambda)\,[\pi - M\{\lambda(y_1-y_0)\}]\,dy_0 = W(y_1,\lambda) \quad (y_1\geqslant 0). \tag{35}$$

This integral equation with y_1 as independent variable can easily be solved by applying the Laplace transformation and using the superposition theorem of the operational calculus (Jeffreys & Jeffreys 1946, pp. 375–376).

Thus we write

$$\left.\begin{aligned}\bar{u}(q,\lambda) &= \int_0^\infty e^{-qy_1}\,u(y_1,\lambda)\,dy_1,\\ \overline{M}_1(q,\lambda) &= \int_0^\infty e^{-qy_1}\{\pi - M(\lambda y_1)\}\,dy_1,\\ \overline{W}(q,\lambda) &= \int_0^\infty e^{-qy_1}\,W(y_1,\lambda)\,dy_1\end{aligned}\right\} \tag{36}$$

and apply the Laplace transformation to equation (35) to obtain

$$\bar{u}(q,\lambda) = \frac{\overline{W}(q,\lambda)}{\overline{M}_1(q,\lambda)}. \tag{37}$$

By substituting from equation (31) in the second of equations (36) we find in fact that

$$\overline{M}_1 = \frac{\pi}{q}\sqrt{\frac{\lambda}{q+\lambda}}, \tag{38}$$

so that equation (37) becomes

$$\bar{u}(q,\lambda) = \frac{q}{\pi}\sqrt{\left(\frac{q+\lambda}{\lambda}\right)}\,\overline{W}(q,\lambda). \tag{39}$$

Now it is interesting to note that the corresponding Laplace transform of $\chi_0(\lambda y_1)$ as defined by

$$\bar{\chi}_0(q,\lambda) = \int_0^\infty e^{-qy_1}\chi_0(\lambda y_1)\,dy_1 \tag{40}$$

is found, after substitution from equation (27), to integrate out in the simple form

$$\bar{\chi}_0(q,\lambda) = \frac{1}{q}\sqrt{\frac{q+\lambda}{\lambda}}, \tag{41}$$

whence equation (39) may be written

$$\bar{u}(q,\lambda) = \frac{q^2}{\pi}\,\overline{W}(q,\lambda)\,\bar{\chi}_0(q,\lambda). \tag{42}$$

Since $W(y_1,\lambda) = 0$ when $y_1 = 0$ from equation (33), $q\overline{W}$ is the Laplace transform of $W'(y_1,\lambda)$, where the dash denotes differentiation with respect to y_1. Hence by the superposition theorem of the operational calculus we can interpret equation (42) and then drop the suffix unity to obtain a solution of our original equation (30) in the form

$$u(y,\lambda) = \frac{1}{\pi}\frac{\partial}{\partial y}\int_0^y W'(y-\mu,\lambda)\,\chi_0(\lambda y)\,d\mu. \tag{43}$$

For a given function $v(y,\lambda)$ the right-hand side of this equation (43) is known by virtue of equations (27) and (33). As in the special case to be considered, however, it will often be simpler to revert to equation (39) and after finding $\overline{W}$ from equations (33) and (36), to interpret direct for u by use of the Bromwich integral.

The general conditions to be satisfied by the function v in order that equation (30) has a valid solution for u of the form obtained, have not been examined since for our physical problem we are concerned only with a particular form for v and we can check our solution *a posteriori*.

Consider now the special case when

$$v(y,\lambda) = K_0\{\lambda(y+\xi)\} \quad (y \geqslant 0,\ \xi \geqslant 0), \tag{44}$$

where ξ is a positive real parameter.

From equations (33) and (31) we then have

$$W(y_1,\lambda) = \frac{1}{\lambda}[M(\lambda\xi) - M\{\lambda(y_1+\xi)\}]. \tag{45}$$

Hence applying the Laplace transform we find, by using the definition of M in equation (31), that

$$\begin{aligned}\overline{W}(q,\lambda) = \int_0^\infty e^{-qy_1}W(y_1,\lambda)\,dy_1 &= \frac{M(\lambda\xi)}{q\lambda} - \frac{2}{\lambda}\int_0^\infty e^{-qy_1}\int_0^{\frac{1}{2}\pi} e^{-\lambda(y_1+\xi)\sec^2\theta}\,d\theta\,dy_1\\ &= \frac{M(\lambda\xi)}{q\lambda} - \frac{2}{\lambda}\int_0^{\frac{1}{2}\pi}\frac{e^{-\lambda\xi\sec^2\theta}}{q+\lambda\sec^2\theta}\,d\theta\\ &= 2\int_0^{\frac{1}{2}\pi}\frac{e^{-\lambda\xi\sec^2\theta}\sec^2\theta}{q(q+\lambda\sec^2\theta)}\,d\theta.\end{aligned} \tag{46}$$

Hence from equation (39) we have

$$\bar{u}(q,\lambda) = \frac{2}{\pi}\sqrt{\left(\frac{q+\lambda}{\lambda}\right)}\int_0^{\frac{1}{2}\pi}\frac{e^{-\lambda\xi\sec^2\theta}\sec^2\theta}{q+\lambda\sec^2\theta}\,d\theta. \tag{47}$$

Before interpreting, we can simplify this by change of integration variable from θ to Y, where

$$\left.\begin{aligned}\tan\theta &= \sqrt{\frac{(q+\lambda)\,Y}{\lambda\xi}},\\ \text{whence}\qquad \sec^2\theta\,d\theta &= \frac{1}{2}\sqrt{\left(\frac{q+\lambda}{\lambda\xi Y}\right)}\,dY,\\ q+\lambda\sec^2\theta &= (q+\lambda)\left(1+\frac{Y}{\xi}\right),\\ \lambda\xi\sec^2\theta &= \lambda\xi+\lambda Y+qY.\end{aligned}\right\} \tag{48}$$

With this substitution equation (47) becomes

$$\bar{u}(q,\lambda) = \frac{1}{\pi}\int_0^\infty \frac{e^{-qY}e^{-\lambda(\xi+Y)}}{\lambda(\xi+Y)}\sqrt{\left(\frac{\xi}{Y}\right)}\,dY, \tag{49}$$

which is in fact of the exact form of the Laplace transformation since the operator q now only appears in the factor e^{-qY} of the integrand. Thus the interpretation of equation (49) may be immediately written down as

$$u(y,\lambda) = I(\lambda y, \lambda \xi), \tag{50}$$

where I is a function of two variables defined by

$$I(\alpha,\beta) = \frac{1}{\pi}\frac{e^{-(\alpha+\beta)}}{\alpha+\beta}\sqrt{\frac{\beta}{\alpha}}. \tag{51}$$

We have now found that equation (50) gives a solution of the integral equation (30) when v is the special case given by equation (44), i.e. we have obtained the relation

$$\lambda\int_0^\infty I(\lambda y_0, \lambda\xi)\, K_0\{\lambda\,|\,y-y_0\,|\}\, dy_0 = K_0\{\lambda(y+\xi)\} \quad (y \geqslant 0,\ \xi \geqslant 0), \tag{52}$$

where I is defined by equation (51).

This relation (52) is the basic result required for the solution of our physical problem and in appendix A is given a direct check of this result by substitution from equation (51) and some manipulation in integration.

We can now return to our physical problem which requires the solution of equation (29). Thus if we put $\xi = 1+y'$ in equation (52), multiply throughout by $\chi_r(y',\lambda)$ and integrate from $y' = 0$ to ∞ we obtain

$$\begin{aligned}\int_0^\infty \chi_r(y',\lambda)\, K_0\{\lambda(y+y'+1)\}\, dy' &= \lambda\int_0^\infty \chi_r(y',\lambda)\int_0^\infty I\{\lambda y_0, \lambda(y'+1)\}\, K_0\{\lambda\,|\,y-y_0\,|\}\, dy_0\, dy' \\ &= \int_0^\infty K_0\{\lambda\,|\,y-y_0\,|\}\int_0^\infty \lambda I\{\lambda y_0, \lambda(y'+1)\}\,\chi_r(y',\lambda)\, dy'\, dy_0. \end{aligned} \tag{53}$$

Hence comparing with equation (29) we obtain

$$\begin{aligned}\chi_{r+1}(y,\lambda) &= \int_0^\infty \chi_r(y',\lambda)\,\lambda I\{\lambda y, \lambda(y'+1)\}\, dy' \\ &= \frac{1}{\pi}\int_0^\infty \chi_r(y',\lambda)\frac{e^{-\lambda(y+y'+1)}}{y+y'+1}\sqrt{\left(\frac{y'+1}{y}\right)}\, dy'. \end{aligned} \tag{54}$$

We have thus obtained a solution of the integral equation (25) in the explicit series form of equation (28) in which the first term is given by equation (27) and each subsequent term is then given from the preceding term by equation (54). This solution is of course for the problem as transformed by the Laplace transformation with respect to time t and we have now therefore to interpret our solution by application of the inverse transformation.

3·3. *Solution for $\psi(y,t)$*

By virtue of equation (17) we have, corresponding to equation (28) a solution for ψ of the form

$$\psi(y,t) = \psi_0(y,t) - \psi_1(y,t) + \psi_2(y,t) - \ldots, \tag{55}$$

where

$$\chi_r(y,\lambda) = \int_0^\infty e^{-\lambda t_0}\,\psi_r(y,t_0)\, dt_0 \tag{56}$$

and conversely by the Bromwich integral

$$\psi_r(y,t) = \frac{1}{2\pi i}\int_L e^{\lambda t}\chi_r(y,\lambda)\, d\lambda. \tag{57}$$

Substituting from equation (56) in equation (54) we obtain

$$\chi_{r+1}(y,\lambda) = \frac{1}{\pi}\int_0^\infty\int_0^\infty \frac{e^{-\lambda(y+y'+1+t_0)}}{y+y'+1}\sqrt{\left(\frac{y'+1}{y}\right)}\psi_r(y',t_0)\,dt_0\,dy', \tag{58}$$

and applying the inverse transformation we interpret this as

$$\psi_{r+1}(y,t) = \frac{1}{\pi}\int_0^\infty\int_0^\infty \frac{\psi_r(y',t_0)}{y+y'+1}\sqrt{\left(\frac{y'+1}{y}\right)}\delta(t-t_0-y-y'-1)\,dt_0\,dy', \tag{59}$$

where $\delta(t)$ is the Dirac function, of Laplace transform unity, which can be defined by (see Carslaw & Jaeger 1941, appendix III)

$$\left.\begin{aligned}\delta(t) &= 0 \quad (t<0)\\ &= \frac{1}{\epsilon} \quad (0<x<\epsilon)\\ &= 0 \quad (x>\epsilon)\end{aligned}\right\} \quad (\epsilon\to 0). \tag{60}$$

We can now perform the t_0 integration in equation (59) to obtain

$$\psi_{r+1}(y,t) = \frac{1}{\pi}\int_0^\infty \frac{\psi_r(y',t-y-y'-1)}{y+y'+1}\sqrt{\left(\frac{y'+1}{y}\right)}dy' \quad (r\geqslant 0,\ y\geqslant 0). \tag{61}$$

The value of ψ_0 as interpreted from equation (27) corresponds of course to the known solution for the half-plane and is

$$\psi_0(y,t) = \delta(t) + \frac{1}{\pi t}\sqrt{\left(\frac{t-y}{y}\right)}H(t-y) \quad (y\geqslant 0) \tag{62}$$

in which the first term $\delta(t)$ is the direct contribution from the incident wave of equation (10) and the second term is the contribution from the diffraction wave sent out from the edge of the half-plane.

Since ψ_0 is zero for negative times we now note that in equation (61), with $r=0$ to give ψ_1, the integrand is zero when $y'>t-y-1$ and consequently if $t<y+1$ the integrand is zero over the whole range and we therefore have

$$\psi_1(y,t) = 0 \quad (t<1+y). \tag{63}$$

If we now assume generally that for a particular value of r,

$$\psi_r(y,t) = 0 \quad (t<r+y), \tag{64}$$

then the integrand in equation (61) will be zero when $t-y-y'-1<y'+r$, i.e. when $t<r+1+y+2y'$ and this will hold over the whole range of integration when $t<r+1+y$ and ψ_{r+1} will then be zero, i.e. equation (64) will hold for $r+1$ if it holds for r. Since by equation (63) it holds for $r=1$ it follows by induction that it is true for all r.

Thus the successive terms in our solution as given by equation (28) each start their contribution at unit time after the preceding term and since in our choice of units this unit time is the time taken for a sound pulse to travel the width of the strip, our solution corresponds, as it should do, to a series of diffraction waves starting from each edge, a wave from any one edge starting a further wave when it reaches the farther edge and so on.

Moreover, we note that by virtue of equation (64) that the solution obtained by stopping at any term ψ_r in equation (55) is in fact the exact solution for $t<r+1+y$. Conversely for

any finite time our solution consists only of a finite number of terms corresponding physically to a finite number of 'reflexions' of the original diffraction wave from each edge. We may also note that by virtue of equation (64) the range of integration in equation (61) is in fact finite.

Equations (55), (62) and (61) thus give an explicit formal solution for $\psi(y,t)$ which is the distribution of $-\partial p/\partial n$ on AB in figure 1 and by symmetry this gives also the distribution on CD in figure 1, whilst on BC we have $\psi=0$ by virtue of the boundary condition.

We have thus obtained the solution for the distribution of ψ over the whole plane $x=0$ and hence from equation (13) the pressure at any point to the rear can be obtained whilst, finally, equation (1) will then give the pressure field in front of the plane $x=0$ containing the strip.

3·4. *Pressure on back of strip*

Although we have thus obtained a complete formal solution of the problem, the evaluation of pressure from equation (13) is likely to be laborious. Further, the pressure is usually of more physical importance than ψ itself and we shall now proceed therefore to obtain an alternative form of solution in which the pressure on the back of the strip is given directly. Thence by use of equations (9) and (1) the whole pressure field can be evaluated without the intermediate calculation of ψ.

Now equation (25) was obtained from equation (13) by putting $X\to 0$ and $p=H(t)$ and then applying the Laplace transformation. For a point $(0,y,0)$ lying on the rear of the strip the same argument can be applied save that the left-hand side of equation (13) is now unknown and if we write

$$\Pi(y,\lambda)=\int_0^\infty e^{-\lambda t}p(y,t)\,dt \quad (-1\leqslant y\leqslant 0), \tag{65}$$

for a point on the rear of the strip, then instead of equation (25) we obtain

$$\Pi(y,\lambda)=\frac{1}{\pi}\int_0^\infty \chi(y_0,\lambda)\,K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0+\frac{1}{\pi}\int_0^\infty \chi(y_0,\lambda)\,K_0\{\lambda\,|\,1+y+y_0\,|\}\,dy_0 \quad (-1\leqslant y\leqslant 0). \tag{66}$$

Since it is easier to think in terms of positive arguments we now write

$$y=-Y \quad (0\leqslant Y\leqslant 1), \tag{67}$$

and we note that in the first integral of equation (66) the argument of K_0 is $\lambda(y_0+Y)$ whilst in the second integral it is $\lambda(y_0+1-Y)$, both these arguments being positive so that modulus signs can be dropped. We can therefore write

$$\Pi(y,\lambda)=\Delta(Y,\lambda)+\Delta(1-Y,\lambda) \quad (0\leqslant Y\leqslant 1), \tag{68}$$

where
$$\Delta(Y,\lambda)=\frac{1}{\pi}\int_0^\infty \chi(y_0,\lambda)\,K_0\{\lambda(Y+y_0)\}\,dy_0 \quad (Y\geqslant 0). \tag{69}$$

If Δ is the Laplace transformation of a function F, so that

$$\Delta(Y,\lambda)=\int_0^\infty e^{-\lambda t}F(Y,t)\,dt, \tag{70}$$

then from equations (65) and (68) we have

$$p(y,t)=F(Y,t)+F(1-Y,t) \quad (0\leqslant -y=Y\leqslant 1), \tag{71}$$

the form of which corresponds of course to the symmetry of the problem.

Although equations (68) and (71) are only valid for the range $0 \leqslant Y \leqslant 1$ we may take the function Δ to be defined by equation (69) for all positive Y and similarly its interpretation F will then be defined for all $Y \geqslant 0$.

Now, using equation (69), we may rewrite the basic integral equation (25) in the form

$$1 = \frac{\lambda}{\pi}\int_0^\infty \chi(y_0, \lambda)\, K_0\{\lambda\,|\,y-y_0\,|\}\, dy_0 + \lambda\Delta(1+y, \lambda) \quad (y \geqslant 0). \tag{72}$$

Also, by mere change of symbols, we may write the fundamental relation (52) in the form

$$\lambda\int_0^\infty I(\lambda Y_0, \lambda Y)\, K_0\{\lambda\,|\,Y_0-y_0\,|\}\, dY_0 = K_0\{\lambda(Y+y_0)\} \quad (Y \geqslant 0,\ y_0 \geqslant 0). \tag{73}$$

If we now write Y_0 for y in equation (72), multiply by $I(\lambda Y_0, \lambda Y)\, dY_0$ and integrate from $Y_0 = 0$ to ∞ we obtain, after change of order of integration and use of equations (69) and (73), the result

$$\int_0^\infty I(\lambda Y_0, \lambda Y)\, dY_0 = \Delta(Y, \lambda) + \lambda\int_0^\infty \Delta(1+Y_0, \lambda)\, I(\lambda Y_0, \lambda Y)\, dY_0 \quad (Y \geqslant 0). \tag{74}$$

This equation can immediately be solved by successive substitutions in the form

$$\Delta(Y, \lambda) = \Delta_0(Y, \lambda) - \Delta_1(Y, \lambda) + \Delta_2(Y, \lambda) - \ldots, \tag{75}$$

where

$$\Delta_0(Y, \lambda) = \int_0^\infty I(\lambda Y_0, \lambda Y)\, dY_0, \tag{76}$$

$$\Delta_{r+1}(Y, \lambda) = \lambda\int_0^\infty I(\lambda Y_0, \lambda Y)\, \Delta_r(1+Y_0, \lambda)\, dY_0. \tag{77}$$

Similarly we can write

$$F(Y, t) = F_0(Y, t) - F_1(Y, t) + F_2(Y, t) - \ldots + (-1)^r F_r(Y, t) + \ldots, \tag{78}$$

where from equations (70) and (75)

$$\Delta_r(Y, \lambda) = \int_0^\infty e^{-\lambda t} F_r(Y, t)\, dt. \tag{79}$$

Equation (77) may then be written, by use of equations (79) and (51), as

$$\Delta_{r+1}(Y, \lambda) = \frac{1}{\pi}\int_0^\infty\int_0^\infty \frac{F_r(1+Y_0, t_0)\, e^{-\lambda(t_0+Y+Y_0)}}{Y+Y_0}\sqrt{\left(\frac{Y}{Y_0}\right)}\, dY_0\, dt_0. \tag{80}$$

If we now interpret this by the Bromwich integral and perform the t_0 integration, in analogous manner to equations (58) to (61) we obtain

$$F_{r+1}(Y, t) = \frac{1}{\pi}\int_0^\infty \frac{F_r(1+Y_0, t-Y-Y_0)}{Y+Y_0}\sqrt{\left(\frac{Y}{Y_0}\right)}\, dY_0 \quad (Y \geqslant 0,\ r \geqslant 0). \tag{81}$$

Similarly, by interpreting equation (76) we obtain

$$F_0(Y, t) = \frac{2}{\pi}\left\{\tan^{-1}\sqrt{\frac{t-Y}{Y}}\right\} H(t-Y) \quad (Y \geqslant 0), \tag{82}$$

which corresponds of course to the pressure on the rear in the half-plane case.

By an analogous argument to that used for the ψ_r functions we can prove by induction that

$$F_r(Y, t) = 0 \quad (t < r+Y) \tag{83}$$

and these $F_r(Y,t)$ terms thus correspond physically to a succession of diffraction waves from the top edge whilst the $F_r(1-Y,t)$ terms in equation (71), using equation (78), correspond to the similar waves from the bottom edge of the strip.

In view of equation (83) it is convenient to take a different origin of time for each F_r and we write

$$\left.\begin{aligned} F_r(Y,t) &\equiv G_r(Y,\tau),\\ \tau &= t-r-Y,\end{aligned}\right\} \tag{84}$$

whence equations (81) to (83) may be written

$$G_0(Y,\tau) = \frac{2}{\pi}\left\{\tan^{-1}\sqrt{\frac{\tau}{Y}}\right\}H(\tau), \tag{85}$$

$$G_{r+1}(Y,\tau) = \frac{1}{\pi}\int_0^{\frac{1}{2}\tau}\frac{G_r(1+Y_0,\tau-2Y_0)}{Y+Y_0}\sqrt{\left(\frac{Y}{Y_0}\right)}\,dY_0, \tag{86}$$

$$G_r(Y,\tau) = 0 \quad (\tau<0), \tag{87}$$

where $r \geqslant 0$, $Y \geqslant 0$ and we can write equation (78) in finite terms for any particular interval of unit time as

$$F(Y,t) = \sum_{r=0}^{n}(-1)^r G_r(Y,t-Y-r) \quad (n+Y \leqslant t < n+1+Y). \tag{88}$$

It may be noted that although $0 \leqslant Y \leqslant 1$ on the strip, the integral in equation (86) implies that any G_r functions must be evaluated for a range of values of $Y>1$ in order to calculate G_{r+1} on the strip.

Our final solution for the pressure at any point on the back of the strip distant Y from an edge is thus given by equations (71), (85) to (88) in explicit form. Thence equation (9) gives the pressure anywhere to the rear and finally equation (1) then gives the pressure field in front.

3·5. *Uniqueness of solution*

To prove the uniqueness of our second form of solution so far as our physical problem is concerned we know first from the initial conditions and since no wave is arriving from the left that the pressure on the back of the strip must be of the form given by equation (71), where the two functions are the same by symmetry, and each term corresponds to effects propagated round one edge. To show that our solution for F is unique we then have from the initial conditions that no effects arrive from the top edge to the point distant Y from it until after time Y (our wave velocity being unity), i.e.

$$F(Y,t) = 0 \quad (t<Y). \tag{89}$$

Now we have shown that the Laplace transform $\Delta(Y,\lambda)$ of F satisfies the integral equation (74), and if we interpret this directly by application of the Bromwich integral we find

$$F_0(Y,t) = F(Y,t)+\frac{1}{\pi}\int_0^{\infty}\frac{F(1+Y_0,t-Y-Y_0)}{Y+Y_0}\sqrt{\left(\frac{Y}{Y_0}\right)}\,dY_0, \tag{90}$$

where F_0 is given by equation (82) as before.

Our solution is one solution of this equation (90) and satisfies also equation (89). If there is a second solution satisfying both these equations then, since the equations are linear, it will differ from our solution by the addition of a solution of the homogeneous integral equation

$$0 = F(Y,t) + \frac{1}{\pi}\int_0^\infty \frac{F(1+Y_0, t-Y-Y_0)}{Y+Y_0} \sqrt{\left(\frac{Y}{Y_0}\right)} dY_0, \tag{91}$$

this additional solution being required also to satisfy equation (89).

Let us assume that the additional solution satisfies

$$F(Y,t) = 0 \quad (t-Y<r), \tag{92}$$

up to some $r \geqslant 0$ and then consider equation (91) for the interval $r \leqslant t-Y<r+1$. For this interval the arguments of F in the integrand of equation (91) will satisfy

$$(t-Y-Y_0)-(1+Y_0) = t-Y-1-2Y_0 \leqslant t-Y-1<r$$

and thus by equation (92) the integral is zero and equation (91) gives zero $F(Y,t)$ for the interval $r \leqslant t-Y<r+1$. Thus if equation (92) is true for r it is true for $r+1$ and since it is true for $r=0$ by equation (89) it follows by induction that it is true for any r. The additional solution F is thus always zero and our solution is therefore the unique solution of the problem. By a similar argument it can be shown that our first form of solution, involving the pressure gradient $\psi(y,t)$ is also unique. In either form, once ψ or F is determined uniquely, the field everywhere is then determined uniquely by equations (1) and either (7) or (9).

Essentially, although the basic integral equation (15) and the resulting equation (25) do not have unique solutions, our solution is made unique by satisfying the condition that there is no incident field from the left and using the fact that effects cannot be propagated round the edges faster than the velocity of sound.

3·6. *Average pressure on strip*

A quantity of physical interest is the force on the strip due to the pulse, which per unit area is the average net pressure $\bar{p}$ given by

$$\bar{p} = \int_0^1 (p_f - p_b)\, dY, \tag{93}$$

where the suffixes refer as before to the front and back of the strip. By use of equations (3) and (10) we may write

$$\bar{p} = 2\int_0^1 (p_i - p_b)\, dY$$
$$= 2(1-\bar{p}_b), \tag{94}$$

where $\bar{p}_b$ denotes the average pressure on the back of the strip.

The average pressure $\bar{p}_b$ can always be obtained by numerical integration of the pressures calculated from our solution for individual points on the back of the strip. In the first two intervals up to $t=2$ we can also obtain direct explicit formulae for this average pressure which lend themselves to accurate evaluation and serve, in particular, as a check on the numerical integration process used for subsequent times.

To derive these expressions it is useful and of some intrinsic physical interest to obtain first a relation between the force on the rear of the strip and the flow to the sides of the strip. Thus, in dimensional units the wave-equation is

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{95}$$

and if this is integrated with respect to y between limits $y=-\alpha$ and $y=\beta$, where α and β are positive constants, we obtain

$$\int_{-\alpha}^{\beta}\frac{\partial^2 p}{\partial x^2}dy+\left[\frac{\partial p}{\partial y}\right]_{-\alpha}^{\beta}=\frac{1}{c^2}\int_{-\alpha}^{\beta}\frac{\partial^2 p}{\partial t^2}dy. \tag{96}$$

Now up to any finite time t_1 (say) we can always choose α and β large enough so that the region of reflected and diffracted waves from the strip lies wholly within $y=-\alpha$ and $y=\beta$. There is then no flow across these planes and for $t\leqslant t_1$ the second term in equation (96) will vanish and thus

$$\frac{\partial^2}{\partial x^2}\int_{-\alpha}^{\beta}p\,dy=\frac{\partial^2}{c^2\,\partial t^2}\int_{-\alpha}^{\beta}p\,dy \quad (t\leqslant t_1). \tag{97}$$

The integrated (or average) pressure between $y=-\alpha$ and $y=\beta$ thus satisfies the plane wave-equation and in particular to the rear of the strip it must behave as a wave travelling in the negative x direction since there are no effects coming from $x=-\infty$. We therefore have

$$\int_{-\alpha}^{\beta}p\,dy=\text{function}\,(ct+x) \quad (x<0, t\leqslant t_1), \tag{98}$$

whence

$$\frac{1}{c}\frac{\partial}{\partial t}\int_{-\alpha}^{\beta}p\,dy=\frac{\partial}{\partial x}\int_{-\alpha}^{\beta}p\,dy=\int_{-\alpha}^{\beta}\frac{\partial p}{\partial x}dy \quad (x<0,\ t\leqslant t_1). \tag{99}$$

This holds also in the limit $x\to -0$ and referring to figure 1, the integrals will then be taken over the back of the strip BC and along portions AB and CD to the sides in the plane of the strip.

But on AB and CD we have $p=p_i$ by equation (2), whilst on BC we have $\partial p/\partial x=0$ by the rigid boundary condition. Hence in the limit $x\to -0$, equation (99) becomes

$$\int_C^B\frac{\partial p_b}{c\,\partial t}dy+\int_D^C\frac{\partial p_i}{c\,\partial t}dy+\int_B^A\frac{\partial p_i}{c\,\partial t}dy=\int_D^C\frac{\partial p}{\partial x}dy+\int_B^A\frac{\partial p}{\partial x}dy \quad (t\leqslant t_1), \tag{100}$$

where limits refer to corresponding points in figure 1 with points D and A subject to the restriction that they lie outside the disturbance due to the strip up to time t_1.

But the incident wave is itself a plane wave travelling in the negative x direction so that we have

$$\frac{\partial p_i}{c\,\partial t}=\frac{\partial p_i}{\partial x}, \tag{101}$$

whence equation (100) may be written

$$\int_C^B\frac{\partial p_b}{c\,\partial t}dy=\int_D^C\frac{\partial}{\partial x}(p-p_i)\,dy+\int_B^A\frac{\partial}{\partial x}(p-p_i)\,dy \quad (t\leqslant t_1). \tag{102}$$

It we now denote by v the particle velocity component in the negative x direction, and by ρ the mass density, then

$$\frac{\partial p}{\partial x}=\rho\frac{\partial v}{\partial t} \tag{103}$$

and inserting this in equation (102) and integrating from -0 to t we obtain

$$\frac{1}{c}\int_C^B p_b\,dy=\rho\int_D^C(v-v_i)\,dy+\rho\int_B^A(v-v_i)\,dy \quad (t\leqslant t_1). \tag{104}$$

The relevant points A and D in figure 1 have been restricted only by the condition that they lie outside the disturbance due to the strip up to time t_1. In fact, v and v_i are identical outside this region so that the integrands on the right-hand side of equation (104) are zero outside the region of disturbance due to reflexion and diffraction at the strip.

The physical interpretation of equation (104) is that the excess rate of total flow to the sides due to the presence of the strip is proportional to the total force (or average pressure) on the rear of the strip.

The preceding relations have been derived generally for any time-variation in the incident pulse. If we now consider our basic pulse defined by equation (10) and revert to our non-dimensional units we can, by use of symmetry and equation (12), write equation (102) in the form

$$\frac{\partial \bar{p}_b}{\partial t} = \int_0^1 \frac{\partial p_b}{\partial t}\, dY = 2\int_0^\infty \{\psi(y,t) - \delta(t)\}\, dy, \tag{105}$$

where we have let the points A and D tend to infinity to cover generally the case of any finite time t_1.

If we now consider alternative derivations of expressions for the average pressure we have first from equation (71)

$$\bar{p}_b = \int_0^1 \{F(Y,t) + F(1-Y,t)\}\, dY$$
$$= 2\int_0^1 F(Y,t)\, dY, \tag{106}$$

whence, by equations (78) and (84), we require the Y integration of the successive G_r functions as given by equations (85), (86) and (87).

Secondly, we can revert to an earlier stage of our solution and seek the Y integration over the strip of the Laplace transforms Δ_r as defined by equations (76) and (77), and then interpret as a final step to obtain the average pressure $\bar{p}_b$ by virtue of equations (78), (79) and (106).

Thirdly, we can use equation (105) and seek the y integration of the ψ_r functions, as defined by equations (61) and (62), with a final integration with respect to time to obtain $\bar{p}_b$.

Fourthly, we note that the Laplace transformation of equation (105) gives

$$\int_0^\infty e^{-\lambda t}\bar{p}_b\, dt = 2\int_0^1 \Delta(Y,\lambda)\, dY$$
$$= \frac{2}{\lambda}\int_0^\infty \{\chi(y,\lambda) - 1\}\, dy, \tag{107}$$

whence we can seek expressions for the average pressure on the back by use of equations (27), (28) and (54) with interpretation as a final step.

By virtue of equation (64) it will be noted that ψ_r will not contribute to equation (105) until $t \geqslant r$. Hence, although the transformed equation (107) will contain contributions from all χ_r functions, the subsequent interpretation for $\bar{p}_b$ will contain no contribution from any particular χ_r until $t \geqslant r$.

For the first interval $0 \leqslant t \leqslant 1$, the evaluation of $\bar{p}_b$ is relatively easy by any of the four preceding methods but for the second interval $1 \leqslant t \leqslant 2$ the fourth method proved the best and we will therefore use it for both intervals.

$0 \leqslant t \leqslant 1$

The only contribution of χ to $\bar{p}_b$ arises from the χ_0 term and we have from equation (27),

$$\int_0^\infty (\chi_0 - 1)\, dy = \frac{2}{\pi}\int_0^\infty \int_0^{\frac{1}{2}\pi} e^{-\lambda y \sec^2\theta} \tan^2\theta \, d\theta \, dy$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\tan^2\theta}{\lambda \sec^2\theta} d\theta = \frac{1}{2\lambda}, \tag{108}$$

whence from equation (107) $$\int_0^\infty e^{-\lambda t}\bar{p}_b \, dt = \frac{1}{\lambda^2},$$

which interprets to give $$\bar{p}_b = t \quad (0 \leqslant t \leqslant 1). \tag{109}$$

In the first interval, therefore, the average pressure (and total force/unit length) on the back of the strip has a simple linear increase with time. This result depends, of course, only on the half-plane solution, and it is perhaps worth noting that, in dimensional units, for a pulse $p_0 H(ct+x)$ incident normally on a half-plane, the total force/unit length on the back at any time is $\frac{1}{2}p_0 ct$, that is, an average pressure of $p_0/2$ on the portion of the half-plane covered by the diffraction wave from the edge.

$1 \leqslant t \leqslant 2$

For this interval we have the same contribution from the $\chi_0 - 1$ term as in the first interval but in addition we have a contribution from χ_1. Now from equations (54) and (27)

$$\chi_1(y,\lambda) = \frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda(y+y'+1)}}{y+y'+1}\sqrt{\left(\frac{y'+1}{y}\right)} dy' + \frac{2}{\pi^2}\int_0^\infty \frac{e^{-\lambda(y+y'+1)}}{y+y'+1}\sqrt{\left(\frac{y'+1}{y}\right)}\int_0^{\frac{1}{2}\pi} e^{-\lambda y' \sec^2\theta}\tan^2\theta\, d\theta\, dy'. \tag{110}$$

Hence, if we integrate χ_1 from $y = 0$ to ∞, interchange orders of integration and write $y = (y'+1)\tan^2\phi$ to substitute ϕ for y as variable of integration we obtain

$$\int_0^\infty \chi_1(y,\lambda)\, dy = \frac{2}{\pi}\int_0^\infty\int_0^{\frac{1}{2}\pi} e^{-\lambda(y'+1)\sec^2\phi}\, d\phi\, dy' + \frac{4}{\pi^2}\int_0^\infty\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} e^{-\lambda y'\sec^2\theta}\tan^2\theta\, e^{-\lambda(y'+1)\sec^2\phi}\, d\theta\, d\phi\, dy'.$$

If we again interchange the order of integration, the y' integration can be performed to give

$$\int_0^\infty \chi_1(y,\lambda)\, dy = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{e^{-\lambda\sec^2\phi}}{\lambda\sec^2\phi} d\phi + \frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{e^{-\lambda\sec^2\phi}\tan^2\theta}{\lambda(\sec^2\theta+\sec^2\phi)} d\theta\, d\phi. \tag{111}$$

We can now use the identity

$$\frac{\tan^2\theta}{\sec^2\theta+\sec^2\phi} = \frac{\sec^2\theta(1+\cos^2\phi)}{\sec^2\theta+\sec^2\phi} - \cos^2\phi$$

to reduce the θ integration in equation (111) to standard integrals and we find

$$\int_0^\infty \chi_1(y,\lambda)\, dy = \frac{2}{\pi\lambda}\int_0^{\frac{1}{2}\pi} e^{-\lambda\sec^2\phi}\sqrt{(\sec^2\phi+1)}\cos^2\phi\, d\phi$$

$$= \frac{1}{\pi\lambda}\int_0^{\frac{1}{2}\pi} e^{-\lambda\sec\eta}(1+\cos\eta)\, d\eta \tag{112}$$

after using the substitution $\sec^2\phi = \sec\eta$.

The total effective contribution to equation (107) in this second interval is thus from equations (108) and (112),

$$\int_0^\infty e^{-\lambda t}\bar{p}_b\,dt = \frac{1}{\lambda^2}-\frac{2}{\pi\lambda^2}\int_0^{\frac{1}{2}\pi} e^{-\lambda\sec\eta}(1+\cos\eta)\,d\eta, \tag{113}$$

whence interpreting,

$$\begin{aligned}\bar{p}_b &= t-\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(t-\sec\eta)\,H(t-\sec\eta)\,(1+\cos\eta)\,d\eta\\ &= t-\frac{2}{\pi}\int_0^{\zeta}(t-\sec\eta)\,(1+\cos\eta)\,d\eta\\ &= t-\frac{2}{\pi}\{\zeta\sec\zeta+\tan\zeta-\zeta-\log(\sec\zeta+\tan\zeta)\}\quad(1\leqslant t\leqslant 2),\end{aligned} \tag{114}$$

where ζ is a parameter defined by $\qquad \sec\zeta = t. \qquad$ (115)

For comparison with an asymptotic solution we require also the net impulse/unit area on the strip which, by virtue of equation (94), is

$$\int_0^t \bar{p}\,dt = 2t-2\int_0^t \bar{p}_b\,dt. \tag{116}$$

Using the parameter ζ it is easy to integrate equation (114) with respect to time and we find for the first two intervals from equations (109) and (114),

$$\int_0^t \bar{p}_b\,dt = \frac{t^2}{2}\quad(0\leqslant t\leqslant 1), \tag{117}$$

$$\int_0^t \bar{p}_b\,dt = t\bar{p}_b-\frac{t^2}{2}+\frac{1}{\pi}\{\zeta\sec^2\zeta+\sec\zeta\,\tan\zeta-\tan\zeta-\log(\sec\zeta+\tan\zeta)\}\quad(1\leqslant t\leqslant 2). \tag{118}$$

Equations (94), (109) and (114) to (118) thus enable both the net force/unit area and the net impulse/unit area exerted on the strip by the pulse, to be evaluated without difficulty to any reasonable degree of accuracy up to time $t = 2$.

3·7. *Asymptotic solution*

Whilst our solution is not difficult to evaluate for the first few intervals, it is obviously not convenient for large times involving the calculation of many diffraction waves and the summation of their effects. In our strip problem with the incident $H(t)$ pulse we know, however, on physical grounds, that the pressures everywhere on the strip will tend ultimately to the unit incident pressure and numerical calculations (§ 5) indicate that such equalization has effectively taken place after about $t = 2{\cdot}5$, i.e. $2{\cdot}5\times$ the time for a sound pulse to travel the width of the strip. Only three diffraction waves have therefore to be evaluated to cover the time to effective equalization. Physically, all that is of much interest occurs in the early stages when our form of solution is eminently suitable and an alternative form of solution suitable for large times is thus only of minor interest in this problem.

The first approximation to an asymptotic solution, analogous to Rayleigh's approximate solution for a sinusoidal wave-train of long wave-length, can be found by the following argument.

If we define

$$p' = \int_0^\infty e^{-\lambda t}(p-p_i)\,dt, \tag{119}$$

where p_i is given by equation (10), then p' will satisfy, in our non-dimensional units,

$$\nabla^2 p' = \lambda^2 p' \tag{120}$$

in space and the boundary condition

$$\frac{\partial p'}{\partial x} = -\int_0^\infty e^{-\lambda t}\frac{\partial p_i}{\partial x}\,dt = -1 \tag{121}$$

on the strip.

Now, in the Laplace transformation the solution for the original function when $t\to\infty$ corresponds to the solution for the Laplace transform when $\lambda\to 0$. Hence the first approximation to our asymptotic solution is obtained by neglecting the right-hand side of equation (120). The solution for p' is then the same as for the velocity-potential in the problem of a lamina of unit width moving broadside-on with unit velocity in an incompressible fluid. From the known solution (Lamb 1932, p. 84) for this problem we can then obtain, in particular,

$$\left.\begin{aligned}\int_0^1 p'\,dY &\to \tfrac{1}{8}\pi \quad \text{on front of strip,}\\ \int_0^1 p'\,dY &\to \tfrac{1}{8}\pi \quad \text{on back of strip,}\end{aligned}\right\}\quad (\lambda\to 0)$$

which we interpret to give

$$\left.\begin{aligned}\int_0^1 p_f\,dY &\to H(t)+\tfrac{1}{8}\pi\,\delta(t),\\ \int_0^1 p_b\,dY &\to H(t)-\tfrac{1}{8}\pi\,\delta(t).\end{aligned}\right\}\quad (t\to\infty) \tag{122}$$

Since $\delta(t)$ is zero except at $t = 0$, equation (122) merely states the trivial result that the average pressures on front and back of the strip tend asymptotically to unit pressure corresponding to eventual complete equalization of the incident pulse round the strip.

However, if we consider the net impulse/unit area averaged over the strip we obtain from equations (93) and (122),

$$\int_0^t \bar{p}\,dt \to \tfrac{1}{4}\pi \quad (t\to\infty). \tag{123}$$

Similar, corresponding to the known semi-circular distribution of velocity-potential over the strip in the incompressible flow problem, we have

$$\int_0^t (p_f - p_b)\,dt = 2\int_0^t (1-p_b)\,dt \to 2\sqrt{\{Y(1-Y)\}} \quad (t\to\infty) \tag{124}$$

for the net impulse/unit area on an element of the strip at distance Y from an edge.

Equations (123) and (124) serve, at least, as a check on numerical calculations of our main solution.

3·8. *Solution for any shape of normally incident plane pulse*

Given the preceding solution for the incident pulse $H(t)$, the solution for any other time-variation in the incident pulse can be immediately obtained by application of the principle of superposition, i.e. in effect by the superposition of the effects of successive infinitesimal increments δp_i. Thus if we distinguish our solution for the pressure at any point due to the $H(t)$ pulse by the suffix zero, then the pressure at this point for an incident pulse $p_i(t)$ striking the strip at time $t = 0$ will be given by

$$p = \int_{-0}^{t} p_0(t-\mu)\,\frac{d}{d\mu}\{p_i(\mu)\}\,d\mu. \tag{125}$$

We write -0 as the lower limit in this equation to indicate that we must include the contribution of an infinite dp_i/dt at $t = 0$ when the incident pulse has a sharp front.

The application of equation (125) can in practice be made in two ways. First, we can apply it to our mathematical form of solution with subsequent numerical evaluation, or secondly, we can apply it directly to our final numerical values for p_0 using numerical integration of equation (125).

The first method reduces to the application of equation (125) to the first term G_0 in equation (88) to obtain a new G_0 from which the subsequent terms are obtained by equation (86) as before. Physically, this corresponds to the fact that the incident field enters directly only into the production of the first diffraction wave from each edge and we can therefore consider any shape of pulse by using the appropriate solution of the half-plane problem to replace equation (85). This first method has advantages if the shape of incident pulse is such that the new G_0 can be expressed simply in terms of known tabulated functions. On the other hand, if the new G_0 has to be evaluated by numerical integration it would appear easier to use the second method of direct numerical integration of equation (125).

By change of integration variable we can write equation (125) in the form

$$p = \int_0^{t+} p_0(\mu)\, \frac{\partial}{\partial t}\{p_i(t-\mu)\}\, d\mu, \tag{126}$$

whence we can also write for a point on the strip or to the rear,

$$p - p_i = \int_0^{t+} \{p_0(\mu) - 1\}\, \frac{\partial}{\partial t}\{p_i(t-\mu)\}\, d\mu. \tag{127}$$

We now note that $p_0 \sim 1$ corresponds to approximate equalization of pressure in our basic case of the $H(t)$ pulse and the calculations described later indicate that such equalization occurs on the strip itself to order 3 % or less by time $t = 2{\cdot}5$. Away from the strip we should expect similar equalization to occur if anything within a shorter time, i.e. $p_0 - 1$, will only be appreciable in equation (127) for a range of 2·5 or less in μ. In general therefore, the application of the principle of superposition by either method will only involve a limited range of integration and calculation for practical purposes.

4. Solution for any incident two-dimensional field

If we have the general case in two dimensions of incident pulses, not necessarily plane, coming from any direction, we can always consider separately the field coming from either side and superpose the solutions. As the two problems are essentially identical, we shall assume the incident field to be arriving from $x > 0$, i.e. from the right in figure 1. By the general theorems of § 2 we can obtain a formal explicit solution for the whole pressure field provided we can derive a solution for either the pressure on the back of the strip or the pressure gradient across AB and CD (figure 1).

Solutions for either of these quantities can be obtained *ab initio* in the general case by a similar analysis to that for our previous special case of normally incident plane pulse. We find first that, if the field is asymmetrical about the centre line of the strip, the application of equations (2) and (13) to a point on AB and to a point on CD (figure 1) leads to two distinct integral equations involving two unknown functions, namely, ψ on AB and ψ on CD. These equations are, however, still linear and can be satisfied simultaneously by assuming a series of the form (55) for each of the two unknown functions. The first term in each series is then a solution of the half-plane problem corresponding to the upper or lower edge respectively, whilst, after application of the Laplace transformation, the $(r+1)$th term in either series is related to the rth term in the other series by an integral equation identical in form with equation (29). Much of the analysis is therefore essentially a repetition of the previous analysis and can, in fact, be avoided if we make the following direct appeal to the physical nature of the problem.

First, although the incident field is not necessarily symmetrical, we still have an inherent symmetry in the strip itself and this can be used to advantage if we introduce two systems of co-ordinates (x, y) and (x, y') one to each edge, by writing

$$y' = -1-y, \quad Y = -y, \quad Y' = -y' = 1-Y, \tag{128}$$

where y, Y as before are measured from the upper edge as origin and y', Y' similarly from the lower edge.

We can now generalize equation (71) to express the pressure on the back of the strip in the form

$$p_b = F(Y, t) + f(Y', t), \tag{129}$$

where F, f refer to diffraction effects arriving from the upper and lower edges respectively.

Corresponding to equation (78) we can then write

$$\left.\begin{aligned} F(Y, t) &= \sum_{r=0}^{\infty} (-1)^r F_r(Y, t), \\ f(Y', t) &= \sum_{r=0}^{\infty} (-1)^r f_r(Y', t), \end{aligned}\right\} \tag{130}$$

in which successive terms represent successive diffraction waves from either edge.

The first term F_0 is then the solution for the incident field striking a half-plane extending downwards from the upper edge of the strip, whilst f_0 is similarly the solution for a half-plane extending upwards from the lower edge of the strip. The F_0 wave will then produce the f_1 wave from the lower edge and this in turn produces the F_2 wave from the upper edge and so on. Thus F_0 leads to the r-even F_r functions and the r-odd f_r functions whilst f_0 similarly produces the r-odd F_r functions and the r-even f_r functions.

Although the initial functions F_0, f_0 depend directly on the incident field, the subsequent production of F_{r+1} by f_r (and f_{r+1} by F_r) is identical with the production of F_{r+1} by F_r in the symmetrical case and we can therefore generalize equation (81) to give

$$\left.\begin{aligned} F_{r+1}(Y, t) &= \frac{1}{\pi}\int_0^\infty \frac{f_r(1+Y_0, t-Y-Y_0)}{Y+Y_0} \sqrt{\left(\frac{Y}{Y_0}\right)} dY_0, \\ f_{r+1}(Y', t) &= \frac{1}{\pi}\int_0^\infty \frac{F_r(1+Y_0, t-Y'-Y_0)}{Y'+Y_0} \sqrt{\left(\frac{Y'}{Y_0}\right)} dY_0. \end{aligned}\right\} \tag{131}$$

The only remaining problem is the determination of F_0 and f_0 in terms of the incident field. Both these initial functions are solutions of the same general problem, namely, the diffraction of any incident field by a half-plane. The solution of this problem is given in appendix B. If we denote the incident field by

$$\left.\begin{aligned} p_i &= F_i(y, t) \quad \text{on } AB \text{ (figure 1)}, \\ p_i &= f_i(y', t) \quad \text{on } CD \text{ (figure 1)}, \end{aligned}\right\} \tag{132}$$

then the solutions for F_0 and f_0 are

$$\left.\begin{aligned} F_0(Y,t) &= \frac{1}{\pi}\int_0^\infty \frac{F_i(y,t-y-Y)}{y+Y}\sqrt{\left(\frac{Y}{y}\right)}\,dy,\\ f_0(Y',t) &= \frac{1}{\pi}\int_0^\infty \frac{f_i(y',t-y'-Y')}{y'+Y'}\sqrt{\left(\frac{Y'}{y'}\right)}\,dy'.\end{aligned}\right\} \tag{133}$$

This completes the solution of the problem for any incident field. As in the special case of the normally incident plane pulse, only a finite number of diffraction waves will be involved up to any finite time after the incident field strikes the strip; the series in equation (130) are thus effectively finite for practical application. For calculation purposes it may be noted that the r-even F_r and r-odd f_r functions form one set and similarly with f and F interchanged we have a second set independent of the first.

If desired, we can alternatively, or in addition, obtain the solution for the pressure gradient ψ across AB and CD in figure 1. We have simply first to generalize equation (55) into two series, one for ψ on AB and one foı ψ on CD. The $(r+1)$th term in either series will then be related to the rth term in the other series by an equation identical in form with equation (61). Finally, the first term in either series will be the solution for ψ in the half-plane problem relevant to the appropriate edge; this solution can be expressed in terms of the incident pressure gradient as shown in appendix B.

5. Numerical calculations. Plane $H(t)$ pulse at normal incidence

5·1. *Scope of calculations*

Calculations of the pressure on the strip have been carried out for representative points up to time $t = 3{\cdot}1$. By symmetry we need only calculate the pressures on half the strip, but since any one diffraction wave G_r is not symmetrical we must in fact calculate it over the whole strip and points $Y = 0{\cdot}1$, $0{\cdot}3$, $0{\cdot}5$, $0{\cdot}7$ and $0{\cdot}9$ were chosen for evaluation.

The function G_0 was tabulated from equation (85) and thence G_1 and G_2 were successively evaluated from equation (86) using numerical integration. The values of these functions are given in tables 1 *a*, 1 *b*, 2 and 3 where it may be noted that for $Y \geqslant 1$ the values were required for the numerical integration and thus had to be tabulated fully with the intervals of 0·1 in Y and 0·2 in τ adopted for this integration. For $0 \leqslant Y \leqslant 1$, however, values of the G_r functions were only required at points at which the final results for pressure were required. The values given in tables 1 *a* and 1 *b* for G_0 are accurate to errors of about three units or less in the fourth decimal place. The values in tables 2 and 3 for G_1 and G_2 are subject to greater errors arising in the numerical integration; independent checks for G_1 using different integration processes indicated that G_1 and G_2 are unlikely to be in error by more than about three units in the third decimal place.

The final values for the pressure on the back of the strip are given in table 4 and plotted in figure 3. Since the interval $\tau = 0{\cdot}2$ is relatively coarse for the initial part of a G_0 wave, the values in table 1 *a* for $\tau < 0{\cdot}2$ were evaluated to give points for figure 3 additional to those provided by the main calculations of table 4.

In table 5 is given the average pressure on the back of the strip. This was calculated from equations (109), (114) and (115) up to time $t = 2$ and thereafter by numerical integration

TABLE 1*a*. VALUES OF $G_0(Y, \tau)$ FOR $0<Y<1$

τ \ Y	0·1	0·3	0·5	0·7	0·9
0	0	0	0	0	0
0·05	0·3918	0·2468	0·1950	0·1660	0·1476
0·1	0·5	0·3333	0·2677	0·2300	0·2048
0·2	0·6082	0·4360	0·3590	0·3125	0·2804
0·4	0·7047	0·5457	0·4645	0·4120	0·3743
0·6	0·7531	0·6082	0·5288	0·4754	0·4360
0·8	0·7837	0·6500	0·5742	0·5212	0·4813
1·0	0·8050	0·6812	0·6082	0·5564	0·5167
1·2	0·8212	0·7047	0·6350	0·5847	0·5457
1·4	0·8340	0·7238	0·6572	0·6082	0·5697
1·6	0·8442	0·7398	0·6755	0·6280	0·5903
1·8	0·8524	0·7531	0·6914	0·6449	0·6082
2·0	0·8601	0·7646	0·7047	0·6595	0·6239
2·2	0·8664	0·7748	0·7168	0·6729	0·6379
2·4	0·8715	0·7837	0·7277	0·6844	
2·6	0·8766	0·7913	0·7366		
2·8	0·8811	0·7983			
3·0	0·8849				

TABLE 1*b*. VALUES OF $G_0(Y, \tau)$ FOR $Y \geq 1$

τ \ Y	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
0	0	0	0	0	0	0	0	0	0	0	0
0·2	0·2677	0·2566	0·2468	0·2380	0·2300	0·2229	0·2164	0·2103	0·2048	0·1997	
0·4	0·3590	0·3454	0·3333	0·3224	0·3125	0·3035	0·2952	0·2875	0·2804		
0·6	0·4195	0·4050	0·3918	0·3798	0·3689	0·3590	0·3498	0·3412			
0·8	0·4645	0·4495	0·4360	0·4234	0·4120	0·4016	0·3918				
1·0	0·5	0·4848	0·4710	0·4583	0·4467	0·4360					
1·2	0·5288	0·5137	0·5	0·4872	0·4754						
1·4	0·5532	0·5383	0·5244	0·5118							
1·6	0·5742	0·5593	0·5457								
1·8	0·5923	0·5775									
2·0	0·6082										

TABLE 2. VALUES OF $G_1(Y, \tau)$

τ \ Y	0·1	0·3	0·5	0·7	0·9	1·0	1·1	1·2	1·3	1·4	1·5
0	0	0	0	0	0	0	0	0	0	0	0
0·2	0·117	0·073	0·058	0·049	0·044	0·042	0·040	0·038	0·037	0·035	
0·4	0·187	0·129	0·106	0·092	0·083	0·083	0·075	0·073	0·070		
0·6	0·241	0·174	0·145	0·127	0·114	0·114	0·105	0·101			
0·8	0·285	0·213	0·178	0·158	0·143	0·142	0·132				
1·0	0·321	0·245	0·208	0·185	0·168	0·167					
1·2	0·352	0·274	0·235	0·210	0·191						
1·4	0·378	0·298	0·258	0·231							
1·6	0·402	0·321	0·279								
1·8	0·422	0·341									
2·0	0·440										

TABLE 3. VALUES OF $G_2(Y, \tau)$ FOR $0<Y<1$

τ \ Y	0·1	0·3	0·5	0·7	0·9
0	0	0	0	0	0
0·2	0·013	0·008	0·006	0·005	0·004
0·4	0·036	0·024	0·019	0·016	
0·6	0·057	0·039	0·032		
0·8	0·077	0·055			
1·0	0·097				

TABLE 4. PRESSURE AT POINTS ON BACK OF STRIP

t	$Y=0\cdot1$	$Y=0\cdot3$	$Y=0\cdot5$	t	$Y=0\cdot1$	$Y=0\cdot3$	$Y=0\cdot5$
0·1	0	0	0	1·7	1·084	1·151	1·154
0·3	0·608	0	0	1·9	1·084	1·102	1·102
0·5	0·705	0·436	0	2·1	1·041	1·056	1·062
0·7	0·753	0·546	0·718	2·3	1·014	1·021	1·026
0·9	0·784	0·920	0·930	2·5	1·006	0·996	0·994
1·1	1·085	1·062	1·058	2·7	0·997	0·985	0·976
1·3	1·078	1·156	1·148	2·9	0·992	0·977	0·978
1·5	1·083	1·153	1·216	3·1	0·993	0·981	0·980

TABLE 5. AVERAGE PRESSURES AND NET IMPULSE/UNIT AREA ON STRIP

t	$\bar{p}_b$	$\bar{p}$	$\int_0^t \bar{p}\,dt$	t	$\bar{p}_b$	p	$\int_0^t \bar{p}\,dt$
0·0	0·000	2·000	0·000	1·9	1·089	−0·179	0·809
0·2	0·200	1·600	0·360	2·0	1·069	−0·138	0·793
0·4	0·400	1·200	0·640	2·1	1·049	−0·098	—
0·6	0·600	0·800	0·840	2·2	—	—	0·774
0·8	0·800	0·400	0·960	2·3	1·018	−0.036	—
1·0	1·000	0·000	1·000	2·4	—	—	0·767
1·1	1·063	−0·127	0·993	2·5	0·999	0·002	—
1·2	1·099	−0·199	0·976	2·6	—	—	0·767
1·3	1·120	−0·241	0·954	2·7	0·988	0·024	—
1·4	1·131	−0·262	0·929	2·8	—	—	0·772
1·5	1·133	−0·266	0·902	2·9	0·983	0·034	—
1·6	1·129	−0·259	0·876	3·0	—	—	0·779
1·7	1·120	−0·240	0·851	3·1	0·986	0·028	—
1·8	1·107	−0·213	0·828	3·2	—	—	0·784

over the strip using Woolhouse's formula (see Whittaker & Robinson 1940, p. 158) and the pressure calculations at the individual points. As an overall check on accuracy this latter process was also applied during the interval $1 \leqslant t \leqslant 2$; comparison with the corresponding values in table 5 derived from the exact equation (114) then indicated errors of 0·003 or less in the average pressure due to the effects in combination of errors in the individual pressures and in the numerical integration by Woolhouse's formula. In the light of this check and of the estimated accuracy of the G_r tables it seems unlikely that the calculations of the pressure at individual points are more than 0·005 in error.

From equation (94) values of the average net pressure on the strip were calculated as given in table 5 and plotted in figure 4. Finally, also in table 5, values of the average net impulse/unit area of strip are given. For $t \leqslant 2$ these values were calculated from equations (115) to (118), whilst for $t > 2$ the values were obtained from equation (116) by using a simple stepped integration (value at mid-point interval) for each successive interval of 0·2 beyond $t = 2$. The net impulse/unit area is plotted against time in figure 5.

5·2. *Discussion*

Turning to the physical interpretation of our results, we note first that the pressure at any point on the back is compounded of successive diffraction waves $G_0, G_1, \ldots$ from both edges of the strip. The even waves $G_0, G_2, \ldots$ give positive contributions whilst the odd waves $G_1, G_3, \ldots$ give negative contributions as seen from equations (71) and (88). In all these waves the only discontinuity in pressure is that in G_0 when both $Y = 0$, $t = 0$ and corresponds to the initial discontinuity at either edge on the arrival of the sharp-fronted incident pulse. Thereafter the pressure is continuous over the strip and at either edge will remain steady at the incident unit pressure. This is to be expected by continuity with the pressure to the sides (i.e. on AB and CD in figure 1) as given by equation (2). It can also be seen from our solution since, by using the substitution $Y_0 = Y\tan^2\theta$ in equation (86), it is easily shown that in the limit $Y \to 0$ we obtain

$$G_{r+1}(0,\tau) = G_r(1,\tau) \quad (r \geqslant 0), \tag{134}$$

whence from equations (71) and (88) the pressure at an edge is given simply by $G_0(0,t)$ which is equal to the incident unit pressure.

From equations (85) and (86) it is also easy to show that

$$G_r(Y,\tau) \propto \tau^{\frac{1}{2}(r+1)}, \quad \text{when } \tau \text{ small, } Y > 0, \tag{135}$$

and thus for given Y and increasing τ the function G_0 has an initial infinite slope whilst the function G_1 has an initial finite slope and all succeeding functions have initial zero slope. The curves for pressure in figure 3 thus have discontinuities in slope at times corresponding to the arrival of a G_0 or G_1 wave from either edge; whereas the former gives an increase to an infinite slope, the latter gives a finite decrease of slope since G_0 is additive and G_1 subtractive in the solution for pressure.

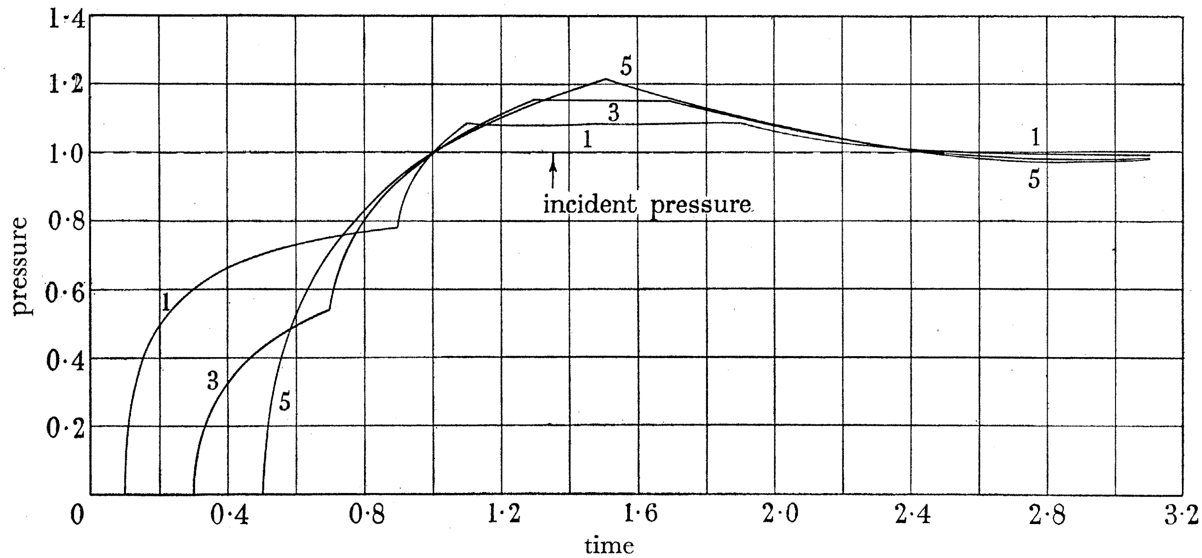

FIGURE 3. Pressure at points on back of strip. 1, $Y = 0{\cdot}1$; 3, $Y = 0{\cdot}3$; and 5, $Y = 0{\cdot}5$.

In figure 3 we see that, from the arrival of the first diffraction wave, the pressure at a point on the back of the strip increases, until at time $t = 1$ it has reached the incident value of unity on all curves. Up to this time, the pressure on the back is simply the addition of the half-plane diffraction waves from either edge, and it is easily seen that when $t = 1$ the angles involved in equation (85) are complementary for the two waves at any point, whence the pressure is exactly unity everywhere on the back. By virtue of equations (3) and (10) we thus see that there is exact equalization everywhere on the strip itself at time $t = 1$, i.e. in the time taken by a sound wave to travel the breadth of the strip.

This equalization is, however, only instantaneous and the pressure thereafter continues to increase until the arrival of the G_1 wave from the nearer edge. We thus have a period during which the pressure on the back of the strip becomes greater than the incident pressure, the excess being as much as 21 % at the centre of the strip where it is greatest. From equation

(3) we have pressures on the front of the strip which are correspondingly less than the incident pressure. We may here note that the pressure-time curves for points on the front of the strip are given simply by reflecting the curves of figure 3 in the line $p = 1$.

Whilst this 'over-equalization' of pressure round the strip is not unexpected, it is interesting to see from figure 3 that the excess pressure on the back lasts for about the same period for all points on the strip, the pressure becoming approximately equalized over the strip at $t = 2{\cdot}5$. Thereafter, the pressure on the back continues to decrease at first and beyond the range of calculations there is presumably a decaying oscillation about the incident pressure $p = 1$. Since the calculations indicate that the first of these oscillations subsequent to $t = 2{\cdot}5$ (approximately) has a maximum deviation of less than 3 % from the incident pressure, any further calculations would demand more accurate evaluations of the G_r functions and would, of course, involve a steadily increasing number of these functions. The only apparent point of possible physical interest in the subsequent oscillations lies in the extent to which they are periodic and the resulting likelihood of any 'resonance' effects with an incident periodic pulse. Such effects are essentially relevant to the problem of an incident sinusoidal train of waves which lies outside the scope of the present paper. It may be

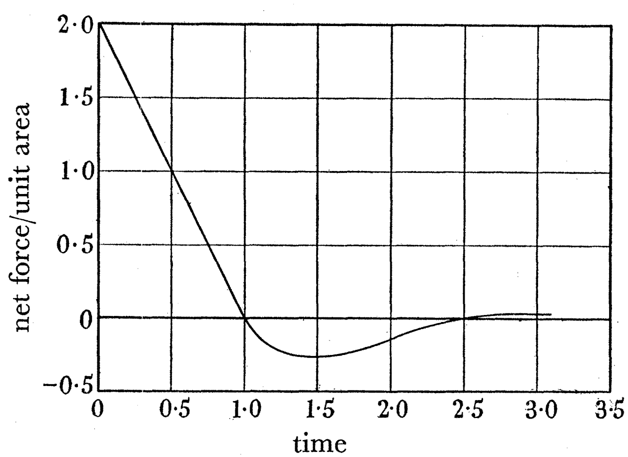

FIGURE 4. Variation of net force/unit area with time.

noted, however, that since a positive swing of 13 % on average pressure during the period $1 \leqslant t \leqslant 2{\cdot}5$ (approximately) is followed by a negative swing of less than 2 %, it seems likely that subsequent oscillations will decay very rapidly with consequent small 'resonance' effects even if the oscillations become essentially periodic.

The average pressure on the back of the strip, as given in table 5, is linear up to $t = 1$ and thereafter follows the same general course as the pressure-time curves for individual points shown in figure 3 save that it is continuous in slope for $t > 0$ when plotted against time.

The most interesting feature of the results is probably the 'over-equalization' phenomenon which shows up clearly in the curve plotted in figure 4 for the variation of the net force/unit area with time. Thus the net force on the strip rises instantaneously to a maximum corresponding to complete reflexion of the incident pulse and then decays linearly to zero at time $t = 1$. It then becomes negative for the longer period to about $t = 2{\cdot}5$, the greatest negative value being 13 % of the initial positive maximum, whilst the impulse in the negative phase is nearly 25 % of that in the initial positive phase. The net impulse exerted on unit area of the strip up to any time is plotted in figure 5 where the curve is parabolic up to its maximum value of unity at $t = 1$ and then decreases to a minimum value of 0·767 at about $t = 2·5$; thereafter the curve is presumably oscillatory about its asymptotic value of $\frac{1}{4}\pi = 0·785$. It will be noted that the minimum at $t = 2·5$ is within $2\frac{1}{2}$ % of the asymptotic value, and similarly integration with respect to time of our results for the pressure at individual points leads to minimum values at about $t = 2·5$, which are within 3 % of the asymptotic values given by equation (124). The impulse calculations thus support the pressure-time curves of figure 3 in suggesting that equalization is effectively reached, to order 3 %, by time $t = 2·5$.

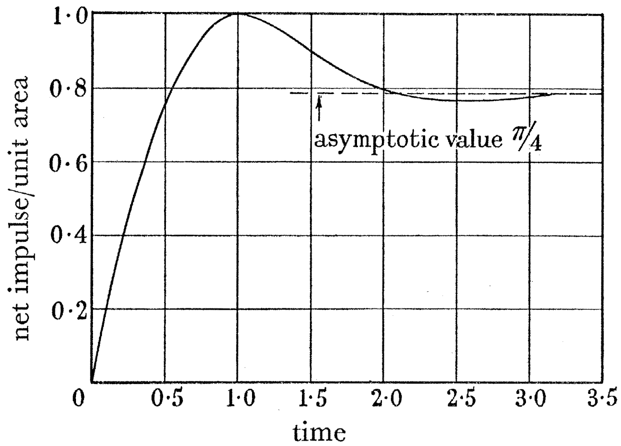

Figure 5. Variation of net impulse/unit area with time.

Summing up, we have an initial period $0 \leqslant t \leqslant 1$ in which pressure is being diffracted round the strip to produce an exact equalization on the strip itself at time $t = 1$. Secondly, we have a period $1 \leqslant t \leqslant 2·5$ (approximately) in which the equalization process may be said to overshoot and produce pressures on the back in excess of incident, up to a maximum excess of 21 % at the centre, with pressures correspondingly below incident pressure on the front of the strip. Thirdly, at about $t = 2·5$, the pressures are again effectively equalized over the strip and thereafter any further oscillations are of order 3 % or less in amplitude with a probable rapid decay.

No calculations have as yet been carried out for pressures at points away from the strip. For such points our solution will involve the numerical integration of equation (9) using the values already calculated for the pressure on the back of the strip. It would undoubtedly be better if the solution away from the strip could be expressed directly in terms of the G_r functions using a special co-ordinate system as in the half-plane problem. This has not been found possible, but in any case it seems doubtful whether calculations of pressure away from the strip would yield any new results of appreciable physical interest. Points on the back of the strip are, in effect, in the deepest shadow, and figure 3 thus indicates the most pronounced effects of diffraction to be expected in the problem.

6. Conclusion

In the present paper has been given an explicit solution and some numerical results for the diffraction of a plane sound pulse of $H(t)$ form (figure 2) incident normally on an infinite strip. The general method of solution appears applicable to a number of two-dimensional diffraction problems involving sound pulses. In particular, it has already been applied to solve the related problems of a slit in a reflecting wall and of a regular grating subjected to the plane $H(t)$ pulse at normal incidence. The analysis and some numerical results for these two problems will be given in further papers.

The extension of the solution to the general case of any two-dimensional pulse field incident on the strip has also been given. This implies that theoretically we can tackle any two-dimensional problem in which a known pulse field is incident on obstacles in the form of strips arranged in arbitrary pattern. Practically, however, it seems fairly certain that the labour of numerical evaluation would be too great, save in relatively simple cases.

For such problems a more comprehensive definition of incident field of pressure can be used to replace the simple definition of § 2. This will be necessary, in general, to allow for the scattered pulses from one strip being rescattered at a second strip and returning (in modified form) to the first strip. Discussion of this more general problem lies outside the scope of the present paper and is deferred to a further paper.

Appendix A

Direct check of equation (52)

To obtain a direct check on the relation (52), where I is given by equation (51) and λ is a complex parameter with positive real part, we consider the left-hand side of equation (52) which, from (51), may be written

$$\lambda\int_0^\infty I(\lambda y_0, \lambda\xi)\, K_0\{\lambda\,|\,y-y_0\,|\}\,dy_0 = I_1+I_2, \tag{136}$$

where

$$I_1 = \frac{1}{\pi}\int_0^y \frac{e^{-\lambda(y_0+\xi)}}{y_0+\xi}\sqrt{\left(\frac{\xi}{y_0}\right)}\,K_0\{\lambda(y-y_0)\}\,dy_0, \tag{137}$$

$$I_2 = \frac{1}{\pi}\int_y^\infty \frac{e^{-\lambda(y_0+\xi)}}{y_0+\xi}\sqrt{\left(\frac{\xi}{y_0}\right)}\,K_0\{\lambda(y_0-y)\}\,dy_0. \tag{138}$$

Now for K_0 we may write

$$K_0\{\lambda\,|\,y-y_0\,|\} = \int_0^{\frac{1}{2}\pi} e^{-\lambda\,|\,y-y_0\,|\sec\theta}\sec\theta\,d\theta, \tag{139}$$

which may be regarded as the definition of K_0 throughout this paper since it was used in introducing K_0 in equation (21) from equation (20) and we do not use any other properties of this Macdonald Bessel function.

Considering I_1 we first change the variable of integration in equation (137) from y_0 to μ, where

$$y-y_0 = \mu \tag{140}$$

and apply equation (139) to obtain

$$I_1 = \frac{1}{\pi}e^{-\lambda(\xi+y)}\int_0^y \frac{e^{\lambda\mu}\sqrt{\xi}}{(\xi+y-\mu)\sqrt{(y-\mu)}}\int_0^{\frac{1}{2}\pi} e^{-\lambda\mu\sec\theta}\sec\theta\,d\theta\,d\mu. \tag{141}$$

We now change the variable of integration in the θ integral by writing

$$\mu(\sec\theta-1) = 2q \tag{142}$$

to obtain

$$I_1 = \frac{1}{\pi}e^{-\lambda(\xi+y)}\int_0^y \frac{\sqrt{\xi}}{(\xi+y-\mu)\sqrt{(y-\mu)}}\int_0^\infty \frac{e^{-2\lambda q}\,dq}{\sqrt{\{q(\mu+q)\}}}\,d\mu, \tag{143}$$

and if we now reverse the order of integration we find that the μ integration can be performed to give

$$I_1 = \frac{2}{\pi} e^{-\lambda(\xi+y)} \int_0^\infty \frac{e^{-2\lambda q} Q}{\sqrt{\{q(q+y+\xi)\}}} dq, \tag{144}$$

where

$$Q = \tan^{-1} \sqrt{\frac{\xi}{\xi+y+q}}. \tag{145}$$

We now consider I_2 and first employ the substitution

$$y_0 = y+\mu \tag{146}$$

in equation (138) and substitute from equation (139) to obtain

$$I_2 = \frac{1}{\pi} e^{-\lambda(\xi+y)} \int_0^\infty \frac{\sqrt{\xi}\, e^{-\lambda\mu}}{(\mu+y+\xi)\sqrt{(y+\mu)}} \int_0^{\frac{1}{2}\pi} e^{-\lambda\mu \sec\theta} \sec\theta \, d\theta \, d\mu. \tag{147}$$

We now substitute q for θ in the second integral, where

$$\mu(1+\sec\theta) = 2q, \tag{148}$$

to obtain

$$I_2 = \frac{1}{\pi} e^{-\lambda(\xi+y)} \int_0^\infty \frac{\sqrt{\xi}}{(\mu+y+\xi)\sqrt{(y+\mu)}} \int_\mu^\infty \frac{e^{-2\lambda q}\, dq}{\sqrt{\{q(q-\mu)\}}} d\mu. \tag{149}$$

We can now reverse the order of integration by use of Dirichlet's formula (see Bocher 1909, p. 4), to give

$$I_2 = \frac{1}{\pi} e^{-\lambda(\xi+y)} \int_0^\infty \frac{e^{-2\lambda q}}{\sqrt{q}} \int_0^q \frac{\sqrt{\xi}\, d\mu}{(\mu+y+\xi)\sqrt{\{(y+\mu)(q-\mu)\}}} dq. \tag{150}$$

We can now perform the μ integration to obtain

$$I_2 = \frac{2}{\pi} e^{-\lambda(\xi+y)} \int_0^\infty \frac{e^{-2\lambda q}}{\sqrt{\{q(q+y+\xi)\}}} (\tfrac{1}{2}\pi - Q)\, dq, \tag{151}$$

where Q is given as before by equation (145).

Hence adding equations (144) and (151) we have

$$I_1 + I_2 = e^{-\lambda(\xi+y)} \int_0^\infty \frac{e^{-2\lambda q}}{\sqrt{\{q(q+y+\xi)\}}} dq, \tag{152}$$

and if we employ a final substitution

$$2q = (\xi+y)(\sec\theta - 1),$$

we find

$$I_1 + I_2 = \int_0^{\frac{1}{2}\pi} e^{-\lambda(\xi+y)\sec\theta} \sec\theta\, d\theta = K_0\{\lambda(\xi+y)\} \tag{153}$$

by virtue of equation (139).

Hence, from equations (136) and (153), we have proved by direct integration the required result

$$\lambda \int_0^\infty I(\lambda y_0, \lambda\xi)\, K_0\{\lambda\,|y-y_0|\}\, dy_0 = K_0\{\lambda(\xi+y)\} \quad (\xi \geqslant 0,\ y \geqslant 0), \tag{52 bis}$$

where I is defined by equation (51). When both ξ and y are zero, our steps are invalid, since integrals involved in the analysis become of infinite magnitude. This special case is, however, trivial, since both sides of equation (52) are then infinite and we understand the equation to mean in this case that the limit of the difference of the two sides is zero when both $\xi \to 0$ and $y \to 0$ through positive values. This is true since our proof will hold throughout the limiting process.

Appendix B

Half-plane subjected to any two-dimensional pulse field

We consider the general problem of a half-plane occupying the y-negative portion of the (y, z) plane and subjected to any type of incident pulse field independent of z and arriving from the right $(x>0)$ of the half-plane. This problem is illustrated in figure 6 where the arrows indicate that the incident field may be arriving from any or several directions on the right.

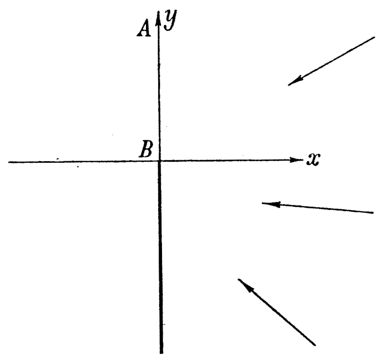

Figure 6

In conformity with our notation for the strip problem we shall write

$$\frac{\partial p}{\partial x} = \psi_0(y, t) \quad (x = 0), \tag{154}$$

whence the boundary condition is

$$\psi_0(y, t) = 0 \quad (y<0). \tag{155}$$

Using equations (2) and (13) we then obtain, with $X \to 0$,

$$p_i(y, t) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{1}{r} \psi_0(y_0, t-r)\, dy_0\, dz_0, \tag{156}$$

in which account has been taken of symmetry in z_0 whilst r is given by equation (16) as before. In this equation (156), the left-hand side is the known incident pressure on AB in figure 6, and we have therefore an integral equation for ψ_0.

To simplify this integral equation by using the Laplace transformation we shall need a relation analogous to equation (18) and we therefore choose our origin of time sufficiently early for

$$\psi_0(y, t) = 0 \quad (t<0), \tag{157}$$

which can always be done if the incident field originates, as in practice, at a finite time prior to its arrival at the edge of the half-plane. The case of a plane wave arriving obliquely from above the half-plane, for which equation (157) is not satisfied, can be covered by considering the limiting form of our solution for a source at a finite distance when this distance increases indefinitely. The preceding choice of our origin of time is dependent essentially on the physical nature of the problem. Thus Huygens's principle in the form of equation (13) states that the pressure to the rear will be affected ultimately by every part of the incident field crossing AB in figure 6, and if the field crosses AB prior to its arrival at the half-plane we must include the contributions of such earlier crossing when we apply the Laplace transformation.

Using Π_i and χ_0 to denote the Laplace transforms of p_i and ψ_0 then equation (156) can be transformed to give, analogous to equation (21), the integral equation

$$\Pi_i(y,\lambda) = \frac{1}{\pi}\int_0^\infty \chi_0(y_0,\lambda)\, K_0\{\lambda\,|\,y-y_0\,|\}\, dy_0 \quad (y \geqslant 0). \tag{158}$$

Similarly, we can apply equation (13) to a point on the back of the half-plane and transform to obtain, analogous to equation (69),

$$\Delta_0(Y,\lambda) = \frac{1}{\pi}\int_0^\infty \chi_0(y_0,\lambda)\, K_0\{\lambda(Y+y_0)\}\, dy_0, \tag{159}$$

where Δ_0 is the Laplace transform of the pressure on the back at a distance Y from the edge.

If we now change the symbols in our basic relation (52) by writing Y for ξ and interchanging y and y_0, we can substitute for K_0 in equation (159); thence by interchange of order of integration and use of equation (158) we obtain

$$\Delta_0(Y,\lambda) = \lambda\int_0^\infty \Pi_i(y,\lambda)\, I(\lambda y, \lambda Y)\, dy. \tag{160}$$

Hence using equation (51) and interpreting we obtain a solution for the pressure p_b on the back of the half-plane in the form

$$p_b = F_0(Y,t) = \frac{1}{\pi}\int_0^\infty \frac{p_i(y, t-y-Y)}{y+Y}\sqrt{\left(\frac{Y}{y}\right)}\, dy. \tag{161}$$

In this form of solution the choice of origin of time is obviously immaterial and we can conveniently change it to the time of arrival of the incident field at the edge of the half-plane. The incident pressure cannot then arrive at any point on AB (figure 6) earlier than time $-y$ so that

$$p_i(y,t) = 0 \quad (t+y<0), \tag{162}$$

whence equation (161) indicates that

$$F_0(Y,t) = 0 \quad (t<Y), \tag{163}$$

corresponding to the propagation of a diffraction wave down the back of the half-plane.

Further, it will be realized that the form of solution given by equation (161) remains unchanged if we proceed from a source at a finite distance in the region $x>0$, $y>0$ to the limiting case of infinite distance corresponding to a plane wave arriving obliquely from above. For this case it can, in fact, be easily checked that equation (161) gives the known solution in the form given by Friedlander (1946).

As discussed in § 2 the general solution for the pressure anywhere can be expressed in terms of F_0 by means of equations (1) and (9). Alternatively, we can use equations (1) and (7) if we know the solution for ψ_0. This latter can be obtained most simply by applying equations (2) and (7) to the problem for $p-p_i$ rather than p. If we write

$$\left.\begin{aligned} \frac{\partial p_i}{\partial x} &= \psi_i(y,t) \quad (y>0) \\ &= \Psi_i(Y,t) \quad (y=-Y<0), \end{aligned}\right\} \tag{164}$$

we then obtain an integral equation relating $\psi_0 - \psi_i$ and Ψ_i which after application of the Laplace transformation becomes an integral equation similar to equation (29) but without the unity term. This can then be solved by use of the basic relation (52) to give when interpreted

$$\psi_0(y,t) = \psi_i(y,t) + \frac{1}{\pi}\int_0^\infty \frac{\Psi_i(Y_0, t-y-Y_0)}{y+Y_0} \sqrt{\left(\frac{Y_0}{y}\right)}\, dY_0. \tag{165}$$

References

Bocher 1909 *Camb. Math. Tract*, no. 10.

Carslaw & Jaeger 1941 *Operational methods in applied mathematics*. Oxford University Press.

Copson 1946*a* *Quart. J. Math.* **17** (65), 20.

Copson 1946*b* *Proc. Roy. Soc.* A, **186**, 100.

Friedlander 1946 *Proc. Roy. Soc.* A, **186**, 322.

Jeans 1925 *Mathematical theory of electricity and magnetism*, 4th ed. Cambridge University Press.

Jeffreys, H. & Jeffreys, B. W. 1946 *Methods of mathematical physics*. Cambridge University Press.

Lamb 1906 *Proc. Lond. Math. Soc.* (2), **4**, 190.

Lamb 1910 *Proc. Lond. Math. Soc.* (2), **8**, 422.

Lamb 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

Macdonald 1902 *Electric waves*. Cambridge University Press.

Magnus 1941 *Z. Phys.* **117**, 168.

Morse & Rubinstein 1938 *Phys. Rev.* **54**, 895.

Sieger 1908 *Ann. Phys., Lpz.*, **27**, 626.

Sommerfeld 1896 *Math. Ann.* **47**, 317.

Sommerfeld 1901 *Z. Math. Phys.* **46**, 11.

Whittaker & Robinson 1940 *The calculus of observations*, 3rd ed. London: Blackie & Son.

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